The Analytic Theory of Point Systems

by

J.D. Bernal

1923
"NOTHING EXISTS EXCEPT ATOMS AND EMPTY SPACE—
ALL ELSE IS OPINION" — DEMOCRITOS

OCCASIONAL PAPER No.1
from the
DEPARTMENT OF CRYSTALLOGRAPHY
BIRKBECK COLLEGE (UNIVERSITY OF LONDON)
MALET STREET, LONDON WC1E 7HX
The Analytic Theory of Point Systems
by
J.D. Bernal
1923
In 1923, Sir William Bragg received the following letter from the mineralogist, Arthur Hutchinson in Cambridge:

Dear Sir William Bragg

25th June 1923

I am venturing to write to you on behalf of a pupil of mine, Bernal by name, whose work on point systems has, I think, been sent to you - Bernal is I think quite a remarkable person; he is a shy, diffident, retiring kind of creature, but something of a genius. He attended my course on Elementary Crystallography and I realized that he was interested and was taking things in quickly. I did not however realize (and he never let on) that he had got so keen that he spent the whole of his next vacation in developing a method of dealing with point systems in the hope that it might be useful in X-ray work! When therefore, he suddenly appeared and deposited on my table a thick type-written MS., rather with the air of a dog bringing a poached rabbit to his master's feet, I was quite amazed - of course I make no pretence of being able to appraise its merit or even its usefulness - still it seemed to me a remarkable effort for an undergraduate in his third year - and Professor H.F. Baker was much interested in it and I believe thinks well of it...

This paper was thus written by John Desmond Bernal (1901-1971) when he was 21. It was submitted as a prize essay to Emmanuel College, Cambridge where Bernal was an undergraduate and earned him the Sudbury Hardyman prize of £30, but it also got him a post with Sir William Bragg and set him on a career in crystallography. The paper was then presented to the Cambridge Philosophical Society on 7th July 1923 as "The Analytic Theory of Crystals" but, although it was accepted, the paper was, on account of its length, never published. One manuscript copy, typed, we believe, by Mrs Eileen Bernal, has circulated in this department for many years, surviving precariously, but we now hasten to publish it in facsimile to avoid further danger of its loss.

Some of the circumstances of the production of this paper are described at length by Professor Dorothy Hodgkin, O.M., F.R.S., who was the earliest of Bernal's students and co-workers during his period in the crystallographic laboratory at Cambridge, in her biographical notice of Bernal (Biographical Memoirs of Fellows of the Royal Society, 26, (1980) which we quote for Prof. Hutchinson's letter and to which reference should be made for details of Bernal's scientific career.
The accompanying article, "X-rays and Crystal Structure" was written by Bernal in 1929 for the 14th Edition of the Encyclopaedia Britannica (and is here reproduced by kind permission of the copyright holders). It clearly follows from Bernal's essay. Bernal also wrote the corresponding article for the 1953 edition of the Encyclopaedia (Vol. 6, pp. 809-829. It ends with the sentences:

"The beating out of metal under the hammer, the brittleness of glass and the cleavage of mica, the plasticity of clay, the lightness of ice, the greasiness of oil, the elasticity of rubber, the contraction of muscle, the waving of hair and the hardening of a boiled egg are among the hundreds of phenomena that had already been completely or partially explained. They were an earnest of the millions of others, old or new, that still had to be explained."

In the period Bernal prepared a further version for the succeeding edition but, regrettably, this was not published because it was too long and had too many illustrations. Evidently the successes of crystallography in explaining the material world and Bernal's enthusiasm had outrun commercial prudence!

J.D. Bernal was elected a Fellow of the Royal Society in 1938 for his work on the elucidation of the structures of biological molecules by the methods of X-ray crystallography and in the same year he succeeded P.M.S. Blackett as Professor of Physics here at Birkbeck College in the University of London. Almost immediately Bernal was drawn into the war but in 1946 the Birkbeck College Research Laboratory for bio-molecular structure was founded in two old houses at 21/22 Torrington Square (now the site of the library of the School of Oriental and African Studies). This was formally opened by Sir Lawrence Bragg on 1 July 1948 for which occasion Bernal wrote: "the central theme of the laboratory is the application of physical methods to the understanding of the structures and reactions of molecules in biological systems". The formal teaching of crystallography (M.Sc. by examination) began at Birkbeck also in 1948 and has continued ever since. In 1964 the Laboratory became the Department of Crystallography (moving into the new extension building and separating from the Department of physics) with Bernal as its first Head and first Professor of Crystallography, but in the same year Bernal suffered the first of a series of strokes. On his retirement in 1968 he was succeeded by C.H. Carlisle and in
1978 by T.L. Blundell, the present occupant of the established chair of crystallography and Head of the Department.

We have invited Professor Rolph Schwarzenberger of Warwick University to give an assessment of this, Bernal's first paper, which, by accident, has remained hitherto unpublished, and we are most grateful for his note which puts the paper into perspective and explains quaternions.

We believe that nothing could be more appropriate as Occasional Paper No. 1 from the Department of Crystallography which Bernal founded to continue the studies which he himself had advanced so much.

Alan Mackay

July 1981
One way to assess the novelty of Bernal's manuscript is to place it side by side with the two books which he himself used:


The two books have in common that they include the listing of the 230 space groups as it was published by Schönflies in 1891 (Nigglı copies Schönflies exactly whereas Hilton changes some nomenclature in translation and alters the order in which the crystal systems occur. Bernal follows Hilton). They differ in that Hilton attempts to give a full proof that there are precisely 230 possibilities whereas Nigglı gives proofs of general results, listings of groups and much additional information about each group (along the lines of the future International Tables). Hilton is more interested in group theory than in crystallography and uses geometrical methods which stem directly from those of Schönflies, whereas Nigglı uses algebra more and deals also with crystal form and with deformations of structure.

It is worth recalling that neither Schönflies nor Fedorov nor Barlow succeeded in getting the number of distinct groups correct at their first attempt: the list of 230 space groups was achieved only as a result of mutual checking between Schönflies and Fedorov. I do not believe that Hilton would have done better, had he not been writing with the list of Schönflies in front of him, because his method is not guaranteed to catch every possible special case of existence or equivalence. Thus Bernal was inspired by Nigglı's more analytic approach to try to make more precise the "simple and geometric" qualitative proof contained in Hilton's book. In this approach Bernal was following very closely in the footsteps of Fedorov who had written over 30 years earlier:

---

*Kristallsysteme und Krystallstruktur*, B.G. Teubner, Leipzig, 1891.


There is no evidence in the manuscript of Bernal having consulted Schönflies directly — in any case Hilton is considerably clearer.
"...here, for the first time, the symmetry of figures is expressed in analytical terms and in this way the theory of symmetry is itself introduced into the realm of analytic geometry. Originally I intended to find analytic terms for the symmetry of finite regular systems. I was promoted to do this by the difficulty of interpreting Sohncke's derivations......an error which remained unnoticed for a long time by the author himself........this could hardly have happened if these systems had been expressed in analytical terms"

(E.S.Fedorov "Symmetry of Finite Figures" 1889 translated by D.Harker 1971)

Similarly Bernal realised that more precise and analytic methods were required for the qualitative listing (Sohncke, Schönflies, Barlow, Hilton) to become useful in the new crystallographic applications. In the remainder of this note I would like to draw attention to three ways in which Bernal notably achieved this aim.

The first important improvement made to Hilton's proof is the decision to work in vector space (i.e. with fixed origin) rather than in affine space. Hilton follows Schönflies in writing, for example, \( A(\alpha) \) to denote a rotation through angle \( \alpha \) about an axis \( A \), or \( S(t) \) to denote a glide transformation with translation component \( t \) lying in the plane of reflection \( S \). But the axis \( A \) and the plane \( S \) need not pass through the origin, which is good for nice qualitative pictures of crystals but bad for precise analytic proofs of the mathematics. Bernal chooses an origin and assigns each affine transformation

\[
X \mapsto qX + c
\]

its linear part \( q \) and its translation component \( c \). The importance of this change for the development of mathematical crystallography is in no way diminished by the fact that, under the steadily increasing influence of modern algebra, many other mathematicians did the same thing quite naturally and most crystallographers do so today.

This leads to the most striking aspects of the manuscript: the influence of modern algebra in the discussion of quadratic forms (in Chapters II and III the symbol \( SXY \) is the scalar product of \( X \) and \( Y \)) and in the use of quaternions throughout. In crystallography the linear part of a symmetry is, of course, a 3x3 matrix determined by coefficients. The fact that \( q \) preserves lengths and angles imposes 6 conditions so only 3 degrees of freedom are left for \( q \). Clearly a more precise analytic approach would benefit from a more economic method of presentation of 3 parameters. It was fairly well known among theoretical physicists and mathematicians that all such linear parts arise from transformations of the form

\[
X \mapsto \pm qXq^{-1}
\]

where \( q \) is a unit quaternion (the correspondence between
unit quaternions and linear parts is actually not one-one, because \(-q\) determines the same transformation as \(q\); this does not cause any confusion in practice. Most readers of this note will be familiar with this fact but, for those who are not, a brief note on quaternions is appended which covers those facts assumed without comment by Bernal in Chapters II and III. Note that the quaternion notation - unlike matrix notation - allows angles of rotation, axis of rotation, plane of rotation to be read off immediately. This is a great advantage when trying to decide whether two groups are or are not equivalent.

The third important improvement is the explicit mention of specialisation: no doubt influenced by the algebraic geometry of the time. This is merely the very simple observation that, for example, a tetragonal lattice is a special case of a orthorhombic lattice. This observation is present but less explicit in Hilton, and explains why Hilton (and so also Bernal) deals with the tetragonal system immediately after the orthorhombic system (Schönflies has the rhombohedral system in between, presumably on the ground that 12 is between 8 and 16).

The effect of these three improvements is that far more facts can be established as general theorems applicable to many crystal systems. Here Bernal is following the example set by Niggli in contrast to Hilton's insistence on a separate discussion for each crystal system. It follows too that the listing of space groups becomes essentially a listing of the relevant translation components which can be handled algebraically. In modern language, Bernal is listing the relevant cocycles to determine a first cohomology group.

In summary, Bernal has rewritten the proof (Sohncke, Schönflies) which he found in Hilton but has brought to it the more analytic attitudes of a quite different crystallographic tradition (Möbius, Fedorov, Niggli). His excellent knowledge of current mathematics yields an improvement upon the treatment of Hilton, 20 years earlier, and comes close to foreshadowing the cohomological work of Zassenhaus, 20 years later.

Ralph Schwarzenberger
April 1981
NOTE ON QUATERNIONS

Any set of 4 real numbers \( a, b, c, d \) can be displayed conveniently as a single quaternion \( a + bi + cj + dk \). The chief advantage of the notation is the ability to multiply using the identities \( i^2 = j^2 = k^2 = -1 \) and \( ij = -ji = k \). Many properties are analogous to those of complex numbers; for example if \( q = a + bi + cj + dk \) then the conjugate \( \overline{q} = a - bi - cj - dk \) satisfies \( \overline{q}q = a^2 + b^2 + c^2 + d^2 \).

For present purposes we need to consider two very special kinds of quaternion \( q \): those for which \( \overline{q} = -q \) (pure quaternions so called because they must necessarily have the form \( q = xi + yj + zk \)) and those for which \( \overline{q} = q^{-1} \) (unit quaternions so called because if \( q = a + bi + cj + dk \) then necessarily \( a^2 + b^2 + c^2 + d^2 = 1 \)). Note that a pure quaternion \( xi + yj + zk \) may be identified with a point \( X = (x, y, z) \) in ordinary three dimensional space, while a unit quaternion \( q = a + bi + jc + kd \) can also be written in the form \( q = \cos \alpha + \sin \alpha (b'i + c'j + d'k) \) for some angle \( \alpha \) where \( b'^2 + c'^2 + d'^2 = 1 \). We assume this form for \( q \) in what follows.

If \( X \) is a pure quaternion then the transformation

\[
\mathcal{Q} : \quad X \mapsto qXq^{-1}
\]

sends \( X \) to another pure quaternion because

\[
\overline{qXq^{-1}} = \overline{q}^{-1}qXq^{-1} = q\overline{X}q^{-1} = -qXq^{-1}
\]

Moreover it is easily shown to preserve lengths and angles. The vector \( X \) is fixed under the transformation if and only if \( qX = Xq \). Inspection shows that this happens if and only if \( X \) is a multiple of \( b'i + c'j + d'k \). Thus \( \mathcal{Q} \) is a rotation about the axis \( (b', c', d') \), and it can be checked that the angle of rotation is \( 2\alpha \).
Similarly the transformation

\[ -\mathbf{q} : \quad \mathbf{X} \longmapsto -\mathbf{qXq}^{-1} \]

is a combination of a reflection and a rotation, that is a symmetry of the second kind.

The power of this notation can be seen when considering composition. If the rotation \( \mathbf{X} \longmapsto q_1\mathbf{Xq}_1^{-1} \) is preceded by the rotation \( \mathbf{X} \longmapsto q_2\mathbf{Xq}_2^{-1} \) then the composite rotation may be represented by the transformation

\[ \mathbf{X} \longmapsto q_2\mathbf{Xq}_2^{-1} \longmapsto q_1(q_2\mathbf{Xq}_2^{-1})q_1^{-1} \]

or precisely the transformation \( \mathbf{X} \longmapsto q_3\mathbf{Xq}_3^{-1} \) where \( q_3 = q_1q_2 \). This is a very much simpler formula for the composition of two rotations than that of Euler and Rodrigues quoted by Hilton.
Preface

I was led to the subject of the present paper by a consideration of the X-ray analysis of crystals. The general uncoordinated and tentative aspect of the methods used seemed to point to an unsuitable geometric groundwork. When Schoenflies, Von Fedorof and Barlow solved the final problem of the 230 types of homogeneous structures there conclusions were purely geometric and there seemed no immediate prospect of applying them to actual crystal structure. As it was only necessary to distinguish one such structure from another, a qualitative basis was sufficient. But this qualitative basis did not meet the needs of X-ray analysis of crystal structure, and the pioneers in this field consequently fell back upon simple and geometric methods which they evolved in the course of their work.

What was wanted, it seemed to me, was an analytic theory in which the structure of a crystal could be represented by a quantitative formula, and the analysis of this structure could be reduced to the solution of certain equations. It occurred to me that the application of simple vector and quaternion calculus would supply the need of experimentalists while at the same time providing yet another solution to the problem of homogeneous structures.
This solution occupies the first six chapters of the paper. It is given in extremely condensed form, and the absence of examples and diagrams may make it difficult to follow the geometric meaning of the various expressions. In this connection references under the heads of Classes and Systems should be made to Hilton's Mathematical Crystallography. Chapters VII and VIII are deductions from the theory which lead to Chapter IX on the X-ray analysis.

Circumstances prevented this Chapter from being as full as was originally intended, especially with respect to the lack of examples and the scant attention paid to the Laue and Hull methods, deficiencies which I intend to make good at a later date.

As the theoretical part has been written independently the absence of references will be understood: the only works to which I am indebted are Hilton (ibid) and Niggli, Analytische Geometrie der Discontinuums, from which I have taken some of the proofs in Chapter III. In Chapter IX I have relied almost entirely on Bragg's X-rays and Crystal Structure, and in view of the forthcoming appearance of the 2nd edition of this work, the elementary nature of the Chapter is perhaps not unfortunate.
CHAPTER I

POINT SYSTEMS

1.1 Point Systems

Consider a system of points in an n dimensional continuum. The position of any point is determined by the vector \( \mathbf{X} \) representing the line drawn from an arbitrary origin \( O \) (not necessarily a point of the system) to the point. This point is called the point \( X \).

The whole system is determined when the origin \( O \) and the vectors corresponding to all the points are given. Thus the system is completely determined by

\[
X, X_1, X_2, \ldots, X_n \text{, written for short } X_r.
\]

\( /X_r \) is called the aspect of the system from \( O \).

2. If \( n \) independent vectors are chosen, i.e. if \( \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n \) can be chosen such that

\[
x_1 \mathbf{A}_1 + x_2 \mathbf{A}_2 + \cdots + x_n \mathbf{A}_n \neq 0 \text{ unless } x_1 = x_2 = \cdots = x_n = 0
\]

then \( X_r \) can always be written

\[
x_n \mathbf{A}_1 + x_n \mathbf{A}_2 + \cdots + x_n \mathbf{A}_n
\]

and the aspect of the system represented by

\[
/X_n \mathbf{A}_1 + x_n \mathbf{A}_2 + \cdots + x_n \mathbf{A}_n
\]

The vectors \( A_1, A_2, \ldots, A_n \) are called the coordinate system or reference vectors.

The scalar coefficients \( x_1, x_2, \ldots, x_n \) are called the coordinates of the point \( X \), and are often used to denote the point \( X \).
**Discontinuous Systems.**

If for two points of the system \( x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots, x'_n \), all the positive scalar differences

\[ x'_1 - x'_2; x'_2 - x'_3; \ldots; x'_n - x'_1; \]

are either 0 or \( e \), when \( e \) is a finite positive scalar, the system is said to be discontinuous. In everything that follows all systems are assumed to be discontinuous.

**Homogeneous Systems.**

If there exists at least \( n \) points whose vectors are independent and from which the aspect of the system is the same, i.e. for which

\( /X_r \) represents the aspect of the system; then the system is called homogeneous.

Let the points

\[ B_1 = b_1 A_1 + b_2 A_2 + \ldots + b_i A_i, \]
\[ B_2 = b_1 A_1 + b_2 A_2 + \ldots + b_i A_i, \]
\[ \vdots \]
\[ B_n = b_1 A_1 + b_2 A_2 + \ldots + b_i A_i. \]

be such a set of points.

then since the system has the same aspect from \( B_1, B_2, \ldots, B_n \), as it has from 0 there is corresponding to any point \( x_1, x_2, \ldots, x_n \) the points

\[ x_1 + b_1 A_1, x_2 + b_2 A_2, \ldots, x_n + b_n A_n. \]

Corresponding to these again are the points

\[ x_1 + 2b_1 A_1, x_2 + 2b_2 A_2, \ldots, x_n + 2b_n A_n. \]

Also the points

\[ x_1 + b_1 A_1 + n_1 b_1 A_1, x_2 + b_2 A_2 + n_2 b_2 A_2, \ldots, x_n + b_n A_n + n_n b_n A_n, \]

in general all the points

\[ x_1 + n_1 b_1 A_1 + n_2 b_2 A_2 + \ldots + n_n b_n A_n, x_2 + n_1 b_1 A_1 + n_2 b_2 A_2 + \ldots + n_n b_n A_n, \ldots, x_n + n_1 b_1 A_1 + n_2 b_2 A_2 + \ldots + n_n b_n A_n. \]
where \( \pm n_r \) can have integral values positive or negative.

Further, the coordinates of any point \( x_1, x_2, x_3, \ldots, x_n \)
become if the point \( n_1 b_1 + n_2 b_2 + \ldots + n_r b_r, n_2 b_1 + n_3 b_1 + \ldots + n_r b_r, \ldots \)
be chosen as origin
\[
-x_1 - n_1 b_1 + n_2 b_1 - \ldots - n_r b_1,
-x_2 - n_1 b_2 + n_2 b_2 - \ldots - n_r b_2,
\ldots,
-x_n - n_1 b_n + n_2 b_n - \ldots - n_r b_n,
\]
and since there is a point with the same coordinates referred to the old
origin, the system must have the same aspect from all the points
\[
/n_1 b_1 + n_1 b_1 + \ldots + n_r b_r, n_2 b_1 + n_2 b_1 + \ldots + n_r b_r, \ldots \ldots n_n b_1 + n_1 b_n + \ldots + n_r b_n
\]
These points form a point system inside the given one and as the
origin was chosen arbitrarily in the first place a set \( \{ A_i \} \) the same aspect
can be found for any origin.

These systems have the property of having the same aspect from every
point of the system. Such systems are called lattices. All lattices
can be written in the form
\[
/n_1 A_1 + n_2 A_2 + \ldots + n_r A_r
\]
when \( \pm n_r \) are integers and \( A_1, A_2, \ldots, A_n \) a set of properly chosen
vectors.

Returning to the original system, now suppose there is another
point \( b_{1m} + b_{2m} + \ldots + b_{nm} \) isognomic with the origin i.e. such that
the system has the same aspect when viewed from these two points;
then the point \( x_1 + n_1 b_1 + n_2 b_1 + \ldots + n_r b_1, x_2 + n_1 b_2 + n_2 b_2 + \ldots + n_r b_2, \ldots \)
and the point \( x_1 + n_1 b_1 + n_1 b_1 + \ldots + n_r b_1, x_2 + n_1 b_2 + n_1 b_2 + \ldots + n_r b_2, \ldots \)
are points of the system. Subtracting, we have the difference of
coordinates \( n_1 b_1 + b_{1m}, n_2 b_1 + b_{2m}, \ldots, n_r b_1 + b_{nm} \)
Now none of these must be infinitesimal and this can only happen
for all integral values of \( n_r \); if \( \frac{b_{1m}}{b_1} \) is commensurate,
extending we have \( \frac{b_{1m}}{b_1} \) commensurate.
Therefore all points isogonomic with 0 may be written in the form
\[ p_n b_n + p_{n-1} b_{n-1} + \ldots + p_1 b_1 + \ldots + p_1 b_1, \ldots \]
where \(p_n\) are rational fractions.

6.5 All the isogonomic points of the system may be written
\[ /p_n B_n + p_{n-1} B_{n-1} + \ldots + p_1 B_1. \]

\(P\) can always be written \(1 + \frac{m'}{n'}\), where \(l, m, n\) are all integers and \(n > m > 0\). Now since \(B\) are isogonomic, \(B_1, B_2, B_n\) are isogonomic points.

The points \(\frac{m_1}{n_1}, \frac{m_2}{n_2}, \ldots, \frac{m_n}{n_n}\) must belong to the system and there must be a finite number \(k\) of such points. Of these \(n\) independent points may be chosen (if \(k > n\) some of the points isogonomic \(B_1, B_2, B_n\) can be taken as well). The number of points with pure fractional coefficients referred to these vectors must be less than before, and continuing the process we must arrive after a finite number of operations to a set of reference vectors in which there is only one point with fractional coefficients and this is the point set of \(0, 0, \ldots, 0\). Such a system is called a primitive.

6.6 All the isogonomic points referred to a primitive set of vectors \(A_1, A_2, \ldots, A_n\) can be expressed in the form
\[ /l_n A_n + l_{n-1} A_{n-1} + \ldots + l_1 A_1, \]
where \(l_n\) take all integral values positive or negative.

This lattice which includes all the isogonomic points of the system with respect to the origin 0 is called the skeletal lattice of the system.
The choice of an elementary set is not unique. There are except for the case \((n=0)\) an infinite number of such sets.

If

\[
A' = l_1 A_1 + l_2 A_2 + \ldots + l_n A_n,
\]

\[
A'' = l_1 A_1 + l_2 A_2 + \ldots + l_n A_n.
\]

is such another set it must be possible to express \(A_1\) in the form

\[
A_1 = l_1' A_1' + l_2' A_2' + \ldots + l_n' A_n.
\]

etc. Where \(l_i'\) are integers.

Now the condition for this is

\[
\begin{bmatrix}
1 & l_1 & l_2 & l_3 & \cdots & l_n
\end{bmatrix} = \pm 1
\]

so that this relation must hold good for the coordinates of any primitive set expressed in terms of any other.

By an extension of this proof it can be seen that if the determinant \(\mathbf{L} = (l_{ij})\) is the set of reference vectors \(A_1, A_2, \ldots, A_n\) is of the \(m\)th order, that is the number of proper fractional points is \(m\) and the highest value of any denominator must also be \(m\).

Component Set.

The general expression for a homogeneous system may now be written

\[
/(x_1 + l_1)A_1 + (x_2 + l_2)A_2 + \ldots + (x_n + l_n)A_n,
\]

\[
/(x_1' + l_1')A_1' + (x_2' + l_2')A_2' + \ldots + (x_n' + l_n')A_n'.
\]

where \(A_n\) is a primitive set. This expression represents every point of the system once and once only if \((x_i', x_i)\) are not all integers for all values of \(r\) (s constant) and of \(a\). Each point \(x_1, x_2, x_3, x_n\) is seen to be accompanied by a lattice of equivalent points; and the number of such lattices \(s\) is called the
Since there are m of such lattices the system is called an m lattice or m point system. The set of points \( x_{1}, x_{2}, x_{3}, \ldots, x_{n} \) is called a component set of the system. One component set corresponds to each point of the skeletal lattice. Instead of \( /x_{1}, x_{2}, x_{3}, \ldots, x_{n}/ \) \((x_{1} + l_{1}), (x_{2} + l_{2}), \ldots, (x_{n} + l_{n})\) might have been chosen as one of the component set, it is consequently possible and sometimes useful to limit the values of the coordinates of the points in a component set. The limits most often employed are (A) \( 0 < x < 1 \) and (B) \(-\frac{1}{2} < x < \frac{1}{2}\). (A) avoids negative coordinates, (B) is more symmetrical.
CHAPTER II.

SYMMETRY

2.1 Characteristics of a Vector set

For every set of \( n \) independent vectors there exists \( n(n+1)/2 \) symmetrical of the type quadratic forms of the type \( Q(X) = q(XY) \); these forms are called the characteristics of the set.

Congruence

Two sets of vectors \( /A \) and \( /A' \) are said to be congruent if

\[ Q(A) = Q(A') \quad \text{and} \quad Q(A_1 + A_2 + \ldots + A_n) = Q(A'_1 + A'_2 + \ldots + A'_n) \]

for all values of \( /A_1, A_2, \ldots, A_n \). Taking these vectors as reference vectors, we have

\[ Q(x_1A_1 + x_2A_2 + \ldots + x_nA_n) = Q(x'_1A'_1 + x'_2A'_2 + \ldots + x'_nA'_n) \]

for all values of \( /x \). From this it follows that if in two systems with congruent reference vectors, there corresponds to every point \( x, x', \ldots, x_n \) in one a point \( x, x', \ldots, x_n \) in the other, then the two systems are congruent and conversely.

Symmetry

If a system is such that two or more aspects of it are congruent though not in general identical, it is said to be symmetrical. If \( O, C, C' \ldots \) are the points from which the system has the same congruent aspects then for any two points \( X \) and \( Y \) there are corresponding points \( C + X', C + Y'; C + X', C + Y'' \ldots \)

where \( X', Y'; X', Y'' \ldots \) are connected by the equations

\[ Q(X) = Q(X') \quad \ldots \ldots \]
\[ q(XY) = q(XY') \quad \ldots \ldots \]
Now $X'$ must obviously be some vector function of $X$ consistent with the above conditions. Writing $X' = f(X)$ we have—if $Y$ is the point corresponding to $X$, i.e. if $Y = X' + C$—we may write as the general form of the $X$ condition of symmetry the identity,

\[ Y = f(X) + C. \]

.4 **Symmetry in three dimensions.**

Leaving at this point the study of point systems in $n$ dimensions we may examine the form which the symmetry function $f(X)$ takes up when $n = 3$ that is in the ordinary space of three dimensions. In this case we may write $Q(X) = X'$ and $Q(XY) = SXY$ and the conditions which $f(X)$ must satisfy become

\[ f(X)' = X' \text{ and } Sf(X)f(Y) = SXY \]

One form of $f(X)$ which satisfies both these conditions is the quaternion transformation

\[ f(X) =qXq' \text{ where } q \text{ is a quaternion} \]

another is the negative quaternion transformation

\[ f(X) =-qXq'. \]

There are no other forms which can satisfy the conditions; for since $f(X)' = X'$, $q$ can always be found such that $f(X) = qXq'$ and similarly $q'$ can always be found such that $f(X)' = q'Xq'$,

then $Sf(X)f(X)' = SQXq'qXq' = SXX'$ if and only if $q' = q$

.5 Thus the general condition of symmetry must always take one or other of the forms $Y = qXq' + G$ and $Y = -qXq + C$. These identities are known as symmetry relations of the first and second sort respectively
The symmetry relation \( Y \equiv qXq' + C \) may always be written in the inverse form \( X \equiv q^{-1}Yq^{-1} + C \) where \( C = qCq' \). Multiplying both of these expressions by \( q \) adding them and dividing the sum by two we arrive at the symmetry relation in its most symmetrical shape,

\[
(Y - \frac{C}{2})q + q(X - \frac{C}{2}) = 0
\]

or \( (Y - \frac{Cq}{2})q + q(X - \frac{C}{2}) = 0 \) for relations of the first and second sort respectively, or more simply still

\[
(Y - C)q - q(X - D) = 0
\]

\[
(Y - C)q + q(X - D) = 0
\]

where \( Cq - qD = 0 \)

In the important particular case in which \( C = 0 \) the relation takes the simplest form of all

\[
Yq = qX + 0.
\]

The unsymmetrical forms are in general more easy to deal with because they involve only one constant vector, and they will be used in all the subsequent work. The other forms were only introduced to show the essential symmetry of the relations.

Combinations and Transformations of Symmetry Relations.

If a system possesses two symmetry relations

(1) \( Y = (-1)^v qXq' + C \)

(2) \( Y = (-1)^v qXq'^{-v} + C \) where \( v \) is an integer odd or even

and in (1) we put \( X = Y' \), we have

(12) \( Y = (-1)^v q'(\{(-1)^v qXq'^{-v} + C\})q'^{-v} + C \)

which reduces to

(12) \( Y = (-1)^{v+v'} q'qXq'^{-v} + (-1)^v q'Cq'^{-v} + C' \)

and since \( qq' \) is a quaternion whose reciprocal is \( q'^{-v} q' \) this is equivalent to another symmetry relation.
This new relation is called the combined relation (12).

The combined relation (21)

\[ (21) \quad Y' = (-1)^q X q'^{q} q' + (-1)^q C q'^{q + c} \]

is in general not the same.

It should be noticed that if (1) and (2) be both relations of the first or both of the second sort their combination is an relation of the first sort; whereas if one is of the first and the other of the second sort their combination is of the second sort.

The combination of one relation with itself n times will always be a relation of the system so that the relation

\[ Y' = (-1)^q X q + c \]

always implies the further relations

\[ Y' = (-1)^q X q^{n + c} \]

where n may have all integral values positive or negative.

The combination (-21)

\[ (-21) \quad Y' = (-1)^q q^q X q'^{q'} q' - (-1)^q q^q C q'^{q'} q' + q C q' + c \]

is called the transformation of (2) by (1).

Reduced Symmetry Relations

In any point system we can always write

\[ X' = X_{m,n} + X' \quad Y' = Y_{m,n} + Y' \]

where \[ X_{m,n} = l A + m B + n C \]

being primitive reference vectors, and \[ m, n \] integers; while

\[ X = p A + q B + r C \]

where \( p, q, r \) are proper fractions though not necessarily rational; similarly for \( Y_{m,n} \) and \( Y' \). The general symmetry relation now takes the form

\[ Y_{m,n} + Y' = (-1)^q (X_{m,n} + X') q' + D \]

writing D for C to avoid confusion

\[ l', m', n' \] are the integers corresponding to \( l, m, n \).

For the skeletal lattice that includes the origin \( X = Y = 0 \)

The relation can therefore be considered as the sum of the two rel-
relations

\[ Y_{\text{m,n}} = (-1)^q X_{i,n} q' + D, \]
\[ Y' = (-1)^q X' q' + D \]

where \( D = D \)

If \( l, m, n \) be put \( = 0 \), it can be seen from the first relation that

\[ D = lA + mb + mC \]

where \( l, m, n \) are integers, now putting \( Y' = D \), instead of \( Y \) we have \( X' = (-1)^q X_{i,n} q' \). In the same way \( D \) can be made to take the form \( pA + qB + rC \) where \( 0 < p, q, r < 1 \). Thus every symmetry relation implies a relation of the type

\[ Y = (-1)^q X p^q \]

for the skeletal lattice, and of the type

\[ Y = (-1)^q X p^q + pA + qB + rC \]

for a component set, or for the whole system. If in addition the system has no relation of the type

\[ Y = (-1)^q X q + D \]

where \( q = q' \)

the above relations are called the reduced symmetry relations of the skeletal lattice and of the system respectively.

---

**Equivalent Points, Simple and Multiple Systems.**

A point system possesses, in general, \( m \) distinct reduced symmetry relations, \( m \) being always finite; or there would be an infinite number of points in each component set. Of these \( m \) relations \( m \) only will be independent all the others can be derived from these \( m \) by combining them in different ways. To any point \( X \) there correspond the points \( X, X, \ldots, X \), to which it is related by the \( m \) relations, also the points \( X, X, \ldots, X \), derived from \( X \) by the \( m \) relations applied \( 2, 3, \ldots, k \) times each (the values of \( k \) will be explained subsequently.) All the points \( X, X, X, \ldots, X \), belong to the same component set, they are called equivalent points.
If these are all the points in each component set, the point system is called simple. If there are other groups of equivalent points \( /X, /Z, \text{ etc.} \) the system is called multiple. The groups \( /X, /Y \text{, etc.} \) may have more but cannot have fewer symmetry elements than the system. The group with the fewest symmetry elements determines the symmetry of the system.

If \( /X \) is such a group the other groups \( /Y, /Z \) can be split up into partial groups \( /X, /Y, \text{ etc.} \) each with the symmetry of \( /X \) and derivable from each other by relations not included in the symmetry of \( /X \).

The whole system is now expressible as: the points \( X, Y, Z \text{, etc.} \) and their equivalent points making up a component set; and the points corresponding to all these points. The points \( X, Y, Z \text{, etc.} \) may be infinite in number and taken together make up a geometrical figure which is in general asymmetrical. For most purposes however it is sufficient to consider simple systems remembering that every point may represent an asymmetrical figure.
3.1 **Symmetry of System and of Skeletal lattice.**

It has been shown already that if a system has a symmetry relation \( Y = (-1)^s q X q^{-1} + D \) the skeletal lattice has the relation \( \sigma Y = (-1)^s q X q^{-1} \). Such relations not involving any constant vector are called rotations. They are in fact representable geometrically by a rotation of the whole system about the axis of \( q \) through twice its angle. Further we must have if \( Y = (-1)^s q X q^{-1} \), is the symmetry group of the system that the skeletal lattice possesses the group of rotations \( Y = (-1)^s q X q^{-1} \). We must therefore examine all the possible classes of groups of rotation that lattices can possess.

.2 **Possible Rotations.**

To begin with we need only consider the form \( Y = a q X q^{-1} \) for since the relation \( Y = X \) holds for all lattices \( Y = q X q^{-1} \) always implies \( Y = q X q^{-1} \) and only those relations of the second kind which correspond to possible relations of the first can exist.

Writing \( q \) in the form \( A^h \) where \( A \) is a vector and \( h \) a scalar the general rotation becomes,

\[
Y = A^h X A^{-h}
\]

The vector \( A \) is called the axis of the rotation.

This relation implies \( Y = A^m X A^{-m} \) where \( m \) is an integer.

But \( Y = A^{2m} X \) i.e. \( Y = X \) is always a relation of the lattice.
Combining we have \( Y = A^{\frac{m+n}{m}} \cdot A^{-\frac{m}{n}} \)

Now \( \frac{m+n}{m} > \frac{m}{m} \) where \( \epsilon \) is a finite positive scalar or \((X - Y)\) would not be a finite vector; \( \theta \) must therefore be rational.

Writing \( \theta \) in the form \( \frac{2k}{k+1} \) where \( k, m, n \) are integers and \( n > m \)

then \( k \theta = \frac{2k}{k+1} \cdot \frac{2}{2} = \frac{2}{k+1} \cdot \frac{2}{2} \)

\( \epsilon - k \theta = \frac{2}{k+1} \cdot \frac{2}{2} \) where \( p \) is an integer > \( m \)

and this must be a possible value of the index of \( A \). Continuing the process we must arrive ultimately at a value \( \frac{2}{k} \) since the value \( \frac{2}{k} \) is inadmissible. The symmetry general rotation now takes the form

\[ Y = A^{\epsilon} X A^{-\epsilon} \]

Such a rotation is called a \( k \) fold axis of symmetry because repeated \( k \) times it leaves the aspect of the lattice as it was to begin with.

### Possible values of \( A \) and \( k \).

From the conditions of symmetry we have

\[ S.AY = S.AX \]

\[ S.A(Y - X) = 0 \]

but \((Y - X)\) must be a point of the lattice for all corresponding values of \( Y \) and \( X \). This shows that in a plane through any point of the lattice perpendicular to any axis of symmetry there is a net of points of the lattice.

### Further let

\[ X_1 = A^4 X A^{-4} \]

\[ X_2 = A^8 X A^{-8} \]

\[ \cdots \cdots \cdots \]

\[ X_t = A^t X A^{-t} \]

\[ X_r = A^r X A^{-r} \]

Adding

\[ X = A^t X_t A^{-t} \]
this is only possible if \( z = pA \), but as the value of the tensor of \( A \) does not affect the symmetry relation, and \( z = pA \), \( z \) is a point of the lattice can always be chosen as a point of the lattice.

Let \( X, X_1, X_2 \) be three points of the lattice satisfying the conditions

\[
S.AX - S.AX_1 = S.AX - 0
\]

\[
X - A^t X A^{-1} = X_1 - A^t X_1 A^{-1}
\]

Rational values of \( p, q, r \) can always be found such that

\[
pX_1 + qX_2 + rX_3 = 0
\]

Multiplying throughout by \( X_3 \) and taking scalar products

\[
pS.X_1 X_3 + qS.X_2 X_3 + rS.X_3 X_3 = 0
\]

\[
(p + r)X_3 \cos \frac{2\pi}{k} + q X_3 \cos \frac{2\pi}{k} = 0
\]

\[
\cos \frac{2\pi}{k} = \frac{q}{p + r}
\]

\( \cos \frac{2\pi}{k} \) is rational, it can only have the values

\[-1, -\frac{1}{2}, 0, \frac{1}{2}, 1 \]

corresponding to the values

\[2, 3, 4, 6, \infty\]

of \( k \), and these must be the only values that \( k \) can have. Excluding the identical relation \( k = \infty \) we have the following theorem.

In any lattice or point system there can only be axes of 2, 3, 4 or 6fold symmetry. Such axes are known as diad, triad, tetrad and hexad axes; respectively. The general rotation must now take one of the particular forms

<table>
<thead>
<tr>
<th>( k )</th>
<th>( Y = A X A^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( Y = A^t X A^{-1} )</td>
</tr>
<tr>
<td>3</td>
<td>( Y = A^t X A^{-1} )</td>
</tr>
<tr>
<td>4</td>
<td>( Y = A^t X A^{-1} )</td>
</tr>
<tr>
<td>6</td>
<td>( Y = A^t X A^{-1} )</td>
</tr>
</tbody>
</table>
Combinations of Rotations

We will first establish some trigonometrical formulae connecting angles between axes and their angles of rotation, i.e., the angles $\frac{2\pi}{\lambda}$

Transform a rotation $Y = B^{t}XB^{-t}$ by the rotation $Y = A^{t}XA^{-t}$

this gives $Y = A^{t}B^{t}XB^{-t}A^{-t} = (A^{t}BA^{-t})X(A^{t}BA^{-t})^{-1} = BXB^{-t}$

$B = A^{t}BA^{-t}$ is called an axis equivalent to $B$. The angle between $B$ and $B'$ is given by

$$\cos \theta = S = S = AB \cdot AB^{-1} = 3 \cdot E (\cos \frac{\pi}{\lambda} + \tilde{A} \sin \frac{\pi}{\lambda})E (\cos \frac{\pi}{\lambda} + \tilde{B} \sin \frac{\pi}{\lambda})$$

(Where $\tilde{A}, \tilde{B}$ stands for versor A B)

$$\sin \phi = \sin \frac{\pi}{\lambda} \sin \phi$$

where $\phi$ is the angle $\tilde{A} \tilde{B}$

Combining the two relations $Y = A^{t}X^{-t}$ and $Y = B^{t}XB^{-t}$, we have $Y = A^{t}B^{t}XB^{-t}A^{-t} = C^{t}XC^{-t}$ where $C = A^{t}B^{-t}$, writing the quaternions in full

$$\cos \frac{\pi}{\lambda} = (\cos \frac{\pi}{\lambda} + \tilde{A} \sin \frac{\pi}{\lambda})(\cos \frac{\pi}{\lambda} + \tilde{B} \sin \frac{\pi}{\lambda})$$

$$\cos \frac{\pi}{\lambda} = \cos \frac{\pi}{\lambda} \cos \frac{\pi}{\lambda} + \cos \theta \sin \frac{\pi}{\lambda} \sin \frac{\pi}{\lambda}$$

Where $\theta$ is the angle $A$ $B$

Consider the two equivalent rotations $Y = A^{t}X^{-t}$, $Y = B^{t}XB^{-t}$. Transforming the second by the first gives $Y = B^{t}XB$ where $B' = A^{t}BA^{-t}$.

A rotation $\phi$ may exist that transforms $B'$ into $A$ and $A$ into $B$.

Then

$$B = C^{t}AC \quad \text{and} \quad B = C^{t}AC^{-t}$$

From these we have

$$B-A = C^{t}(A-B)C$$

The angle between $(B-A)$ and $(A-B) = \frac{\pi}{\lambda}$

$$\cos \frac{\pi}{\lambda} = -S(B-A)(A-B)$$

$$S(E^{t}B^{t}A^{t}B^{t}A^{-t}B^{-t})$$

$$= -1 + 8 \cos \theta \cos \frac{\pi}{\lambda} - \cos 2 \theta \sin \frac{\pi}{\lambda}$$

where $\theta$ is the angle between $A$ and $B$.

$$= -1 \cdot \sin \frac{\pi}{\lambda} (1 + \cos \theta)$$
That is \( \sin \frac{\pi}{m} - \sin \frac{\pi}{n} \cos \frac{\pi}{z} \) or \( \cos \frac{\pi}{z} = \frac{\sin \frac{\pi}{m}}{\sin \frac{\pi}{n}} \)

Giving \( m \) and \( n \) all allowable values for which \( m > n \) we obtain all the possible values of \( \cos \frac{\pi}{z} \) which are shown in the following table

<table>
<thead>
<tr>
<th></th>
<th>4 : 3 : 2</th>
<th></th>
<th>6 : 1 : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{3}}</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1 : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{3}}</td>
<td></td>
<td>3 : 1 : \frac{1}{\sqrt{3}}</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2 : 1 : 1</td>
<td></td>
<td>2 : 1 : 1</td>
<td></td>
</tr>
</tbody>
</table>

From this we see that

6 fold axes cannot exist in different directions

4 fold axes must be at right angles

3 fold axes must intersect at angles of \( \cos \frac{\pi}{z} \) or the complementary angle of \( \cos \frac{\pi}{z} \)

2 fold axes when equivalent must intersect at angles of \( \frac{2\pi}{3} \) or \( \frac{\pi}{3} \).
Classes of Rotation groups: First Sort.

We are now in a position to consider the various classes into which groups of rotations of the first sort may be divided. These classes will be designated for convenience of reference by their Schoenflies symbols.

Groups containing one rotation only. General type \( Y = A^x X A^z \)

There are four classes \( k = 2, 3, 4, 6 \)

symbol \( C_2, C_3, C_4, C_6 \).

Groups with one \( k \) fold and 2 fold axes only.

General type \( Y = A^x X A^z \). \( B_k, B'_k \) are equivalent axes. Any two of the axes \( A, B_k, B'_k \) can be taken as the independent rotations of the group.

We have by substituting in transforming \( A \) by \( B \) we must obtain \(-A\) or they would be two \( k \) fold axes which is inadmissible in this type. The angle \( \phi \) between \( A \) and \( B \) must be \( \frac{\pi}{k} \), and substituting in \( 3 \cdot 4! \) we have

\[
\sin \theta = \sin \frac{\pi}{k}, \quad \theta = \frac{\pi}{n}
\]

If \( k \) is of the form \( 2h \). \( B_k, B'_k \) there are only \( h \) distinct axes equivalent to \( B \). In this case also the combined relations \( A^x B \) give rise to \( h \) diad axes distinct \( B_k, B'_k \) distinct from \( B, B_k, B'_k \).

Now \( A^x B \cdot A^x B = B \), so that \( B \) can be put in the form \( B + B \).

If \( k = 3 \) the combined relation \( A^x B \cdot B \) so that there are only three equivalent diad axes and no more.
Here again there are four classes: \( n = 2 \ 3 \ 4 \ 6 \)

symbol \( D_1 \ Q \ D_2 \ D_3 \ D_4 \ D_6 \)

\( Q \) Rotations \( Y = A A^t \quad Y = B B^t \quad Y = C C^t \)

With the conditions \( S.BC = S.AC = S.AB = 0 \)
\( S.ABC = -1 \)
\( S.AB = S.AC = S.BC = 0 \)
\( S.\overline{A}C = S.\overline{A}B = 0 \)
\( B = C \)

\( D_4 \) \( Y = A^\frac{1}{4} A^\frac{1}{4} \quad Y = B B^t \quad Y = C C^t \quad Y = (B+C)(B+C)^t \)

\( S.AB = S.AC = S.AD = 0 \)
\( S.\overline{A}D = S.\overline{A}B = S.DC = -\frac{1}{7} \)
\( B = C = D^2 \)

\( D_5 \) \( Y = A^\frac{1}{4} A^\frac{1}{4} \quad Y = B B^t \quad Y = C C^t \quad Y = D^t \)

\( S.\overline{A}D = S.\overline{A}B = D^2 \)

\( D_6 \) \( Y = A^\frac{1}{4} A^\frac{1}{4} \quad Y = B B^t \quad Y = C C^t \quad Y = (C+D)(C+D)^t \)

\( B + C = D = 0 \)

Same as \( D_2 \)

More than one \( k \) fold axes \((k > 2)\). There are two classes:

- \( T \) with triad and diad axes; \( O \) with tetrad, triad and diad axes.

Both have a set of four triad axes \( T_1, T_2, T_3, T_4 \) which are all equivalent and must by \( \frac{343}{4} \) and considerations of symmetry fulfill the conditions

\[ \overline{T} \ 7/3 \ T_1 = T_1 \ T_2 = T_2 \ T_3 = T_3 T_4 = T_4 T_1 = T_1 T_2 T_3 T_4 = 0 \]

In class \( T \), only the three diad axes \( A, B, C \) related to \( T_1, T_2, T_3, T_4 \)

by the rotations (written in their symmetrical form)

\[ A T_1 - T_2 A = A T_1 - T_2 A = A T_1 - T_2 A = A T_1 - T_2 A = 0 \]
\[ T_1 = A T_1 - T_2 A = A T_1 - T_2 A = A T_1 - T_2 A = A T_1 - T_2 A = 0 \]

for \( T_1 \) and \( T_2 \), from these we find \( S.BC = S.AC = S.AB = 0 \)

\[ A^2 = B^2 = C^2 \]

also \( T_3 = A + B + C \); \( T_4 = A + B - C \); \( T_5 = A - B + C \); \( T_6 = A - B - C \). We have therefore for

\( T \) Rotations \( Y = A A^t \quad Y = B B^t \quad Y = C C^t \quad Y = (A + B + C) X(A + B + C)^t \quad Y = (A + B + C) X(A + B + C)^t \quad Y = (A + B + C) X(A + B + C)^t \)
Class 0 differs by having $A'B'C$ as tetrad axes and in consequence has four more triad axes which by virtue of the relations $T_3 = A^*_T, A^*_{\cdot}$ are the inverses of the original four. Also for the same reasons as $D_4$ it has three pairs of diad axes of the type $B+C$ or $T_4,-T_4$ are.

In all there are thirteen rotations any two of which may be chosen as independent if they are of different kinds.

$$
\begin{align*}
Y &= A^*_T X A \\
Y &= B^*_T X B \\
Y &= C^*_T X C \\
Y &= (A+B+C)^ bulk \times (A+B+C)^ bulk \\
Y &= (A+B+C)^ bulk \times (A+B+C)^ bulk \\
Y &= (A+B+C)^ bulk \times (A+B+C)^ bulk \\
Y &= (B+C) \times (B+C)' \\
Y &= (C+A) \times (C+A)' \\
Y &= (C-A) \times (C-A)' \\
Y &= (A+B) \times (A+B)' \\
Y &= (A-B) \times (A-B)'
\end{align*}
$$

There are no other classes of rotation groups of the first sort.

By $343$ there are no other arrangements of triad and tetrad axes possible than those of $T$ and $O$ and a lattice can only possess one hexad axis.
Classes of Rotation groups: Second sort

The general rotation of the second sort is
\[ Y = A^{\pm}X^A \]

Combined with another relation of the second sort it gives rise to one of the first sort, so we need only consider classes containing second sort operations only, or those formed by combining with one of the classes of the first sort one rotation of the second sort.

Further since the effect of combining \( Y = A^{\pm}X^A \) and \( Y = -A^{\pm}X^A \) to form \( Y = A^{\pm}X^A \) is the same as that of combining \( Y = A^{\pm}X^A \) and \( Y = -X \)
only we need consider the combinations of \( Y = -X \) with classes of the first sort and the combinations of \( Y = -AXA^{-1} \) with classes of the first sort which have no axis in the direction \( A \). \( Y = -AXA^{-1} \) is the only second sort rotation which can be combined in this way because all the others include a rotation of the first sort.

Second sort operations alone. General type \( Y = A^{\pm}X^A \)

There are five classes \( K = \infty \ 2 \ 3 \ 4 \ 6 \)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>( C_i )</th>
<th>( C_s )</th>
<th>( C_{2i} )</th>
<th>( C_4 )</th>
<th>( C_{3h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_i )</td>
<td>( Y = -X )</td>
<td>This is called a centre of symmetry.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C_s )</td>
<td>( Y = -AXA^{-1} )</td>
<td>&quot; &quot; &quot; a plane of symmetry.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C_{2i} )</td>
<td>( Y = -A^{\pm}X^A )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C_4 )</td>
<td>( Y = -A^{\pm}X^A )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C_{3h} )</td>
<td>( Y = -A^{\pm}X^A )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Except in the case where the axis to which \( A \) is perpendicular is a diad axis. We have therefore also to consider the results of combining second class operations rotations with perpendicular diad axes.
Combination of $Y = -X$ with first sort, nine classes

Ten classes corresponding to $C_2, C_3, C_4, C_5, C_6, D_3, D_4, D_5, T, O$

Symbol $C_2, C_3, C_4, C_5, C_6, D_3, D_4, D_5, T, O$

One of these $C_i$ belongs also to the last set so that there are really only nine classes.

For $C_2$
- Relations in addition to those of $C_2$
- Conditions as for $C_2$

For $C_3$
- $Y = -X$
- $Y = -AXA^{-1}$

For $C_4$
- $Y = -X$
- $Y = -AXA^{-1}$
- $Y = -A^2XA^{-1}$

For $C_5$
- $Y = -X$
- $Y = -AXA'^{-1}$
- $Y = -A^2XA'^{-1}$
- $Y = -A^3XA'^{-1}$

For $C_6$
- $Y = -X$
- $Y = -AXA'^{-1}$
- $Y = -BXB'$
- $Y = -CX'C'$

For $D_3$
- $Y = -X$
- $Y = -AXA^{-1}$
- $Y = -BXB'$
- $Y = -CX'C'$
- $Y = -DXD'$

For $D_4$
- $Y = -X$
- $Y = -AXA'^{-1}$
- $Y = -A^2XA'^{-1}$
- $Y = -BXB'$
- $Y = -CX'C'$
- $Y = -(B+C)X(B+C)^{-1}$
- $Y = -(B+C)X(B+C)^{-1}$

For $D_5$
- $Y = -X$
- $Y = -AXA'^{-1}$
- $Y = -A^2XA'^{-1}$
- $Y = -BXB'$
- $Y = -CX'C'$
- $Y = -DXD'$
- $Y = (C+D)X(C+D)^{-1}$
- $Y = (D+B)X(D+B)^{-1}$
- $Y = (B+C)X(B+C)^{-1}$
Relations in addition to those of $T$

conditions as for $T$

\[\begin{align*}
Y = -X \\
Y = -AXA' \\
Y = -BXX' \\
Y = -CXX' \\
Y = -(A+B+C)X(A+B+C)' \text{ ETC.}
\end{align*}\]

$C_h = \ldots = 0$

\[\begin{align*}
Y = -X \\
Y = -AXA' \\
Y = -BXX' \\
Y = -CXX' \\
Y = -(B+C)X(B+C)' \text{ ETC.}
\end{align*}\]

\[Y = -A^\frac{1}{k}XX' \text{ ETC.} \]

\[Y = -(A+B+C)^\frac{1}{k}X(A+B+C)' \text{ ETC.}\]

68. Combination of $Y$ $BBB$ with rotations $Y$ of the first sort

Four classes corresponding to $O_2$, $C_1$, $C_2$, $C_2$

Symbol $O_{2r}$ $C_{1r}$ $C_{2r}$ $C_{2r}$

$C_{2r}$ Relations in addition to those of $C_2$

\[\begin{align*}
Y = -BXX' \\
Y = -CXX'
\end{align*}\]

$C_{1r}$

\[\begin{align*}
Y = -BXX' \\
Y = -CXX'
\end{align*}\]

$C_{2r}$

\[\begin{align*}
Y = -(A+B)X(C+D) \\
Y = -(D+B)X(D+B)' \\
Y = -(B+C)X(B+C')
\end{align*}\]

$C_{4r}$

\[\begin{align*}
Y = -BXX' \\
Y = -CXX'
\end{align*}\]

The only other class of the first sort with no axes perpendicular
to a $k$ fold axis ($k > 2$) is $T$. From this we have therefore one

class $T_d$ Relations in addition to those of $T$

\[\begin{align*}
Y = -(B+C)X(B+C)' \\
Y = -(B-C)X(B-C)' \\
Y = -(C+A)X(C+A)' \\
Y = -(B-C)X(C-A)' \\
Y = -(A+B)X(A+B)' \\
Y = -(A-B)X(A-B)'
\end{align*}\]
Lastly we have to consider the classes which are the result of combining a diad axis \( Y = XX \) with a rotation of the second sort \( Y = -A^X \), where \( S.AB = 0 \) and \( k > 2 \). There are three classes for \( k = 3, 4, 6 \) as \( Y = -A^X \) itself possesses a centre of symmetry \( D_4 \) must have one and we have already dealt with it. There remain two new classes \( D_4 \) whose rotations are:

\[
\begin{align*}
Y &= -A^X = A^X \quad \text{with the conditions} \\
Y &= XX \\
Y &= AB = S.AB = S.BC = 0 \\
Y &= -(B+C)(B+C) = B = C \\
Y &= -(B+C)(B+C) = B = C = D \\
Y &= -(B+C)(B+C) = B+C = 0
\end{align*}
\]

In both these cases can be expressed alternatively with rotations \( Y = -AXA^* \) etc and \( Y = (B+C)(B+C) \) etc. This is of some importance subsequently.

We have now derived all the classes of both sorts into which groups of rotations can be divided. There are thirty two in all, if we include that class with no rotation but the identity relation \( Y = X \). They are divided for reasons that will appear later into five "systems".

I. Triclinic system

\[ C, C_i \]

II. Monoclinic system

\[ C_1, C_2, C_4 \]

III. Orthorhombic system

\[ C, D_2, D_4 \text{ or } Q \]

IV. Tetragonal system (a) tetrad axis last sort

\[ C_4, C_{4r}, D_4, D_4 \]

(b) " 2nd "

\[ C, D_4 \]

V. Rhombohedral system

\[ C_3, C_{3i}, C_{3r}, D_3, D_3' \]
VI. Hexagonal system (a) Hexad axis 1st sort $C_6, C_{6h}, C_{6v}, D_6, D_h$.
(b) " " End " $C_{3A}, D_{3h}$.

VII. Regular system $T, T_h, T_d, O, O_h$.

The classes $C_i, C_{2A}, C_{2h}, D_{2h}, D_{3h}, A_{0h}$ are called holohedral.
Holohedral classes have all the rotations of both sorts that are in the other classes of the same system.
The classes $C_1, C_2, D_2, D_3, D_4, O$ are called holoaxial. They contain all the rotations of the first sort that are in the system.
The classes $C_2, C_3, C_4, C_5, C_{3v}, C_{3r}, C_{4v}, C_{4r}, T, T_d$ are called hemimorphic. They all contain one sided axes, that is axes such as $A$, to which there corresponds no axis $-A$. 
CHAPTER IV.

LATTICES

0 Types of symmetrical lattices.

We are now in a position to examine the number of types to which any lattice must correspond. The symmetry of any lattice must be that of one of the thirty-two classes, but the following considerations show that the choice of these is limited.

01 In a lattice containing a k-fold axis A, (k > 2) let B be a primitive vector perpendicular to A such that no vector B also perpendicular to A exists making \(B > B'.\) Then if \(B = A^\frac{k}{k} B^\frac{k}{k}, B = A^\frac{k}{k} B^\frac{k}{k}\) the pairs \(B,B'_1\) and \(B,B'_2\) must be primitive vectors for the net perpendicular to A. Also since \(B = B B^r\) B and likewise \(B_1\) and \(B_2\) must be diad axes for the net, further \(A, B, B, -A, B, B,\) are congruent reference systems which are related by a rotation about B so that B must be a diad axis of the matrix lattice. From this we have the theorem.

There are diad axes perpendicular to every triad, tetrad, and hexad axis of a lattice.

02 Every lattice has a centre of symmetry \(Y = -X.\) Combining this with the symmetry axes of the lattice we have the theorem.

Perpendicular to every diad, tetrad, and hexad axis of \(\infty\) lattice there is a plane of symmetry and every triad axis is a plane of the second sort. From these two theorems we can see that the only classes of symmetry to which lattices can belong are the holohedral classes \(C_i, C_{i4}, C_{44}, D_{44}, D_{44}, D_{44}, O_h.\)
Let us begin with lattices whose class of symmetry is that of \( C_i \). This is the most general type and the choice of a suitable reference set is arbitrary so that we can always choose three primitive vectors \( a, b, c \) and write the formula of the lattice

\[
\Gamma_{tr} = \{ la + mb + nc \}
\]

Here and in the subsequent work the small letters \( a, b, c \) (\( d \)) are used exclusively to denote reference vectors; capitals \( A, B, \) being used in the general expressions for axes. The letters \( 1, m, n, (0) \) will always represent integers, while \( p, q, r, (s) \) and \( u, v, w, (t) \) stand for rational fractions. The rotations of the various lattices are not given here; they are those of the class of symmetry to which the lattice belongs.

\( C_{4h} \)

\( a \) can here be chosen to be the one diad axis while \( b \) and \( c \) can be chosen arbitrarily so as to be primitive vectors in the net perpendicular to \( a \). The formula of the lattice is now \( \Gamma_{pa+qb+rc} \) as \( a, b, c \) are not necessarily primitive vectors for the lattice. Now if \( p, q, r \) is a point of the lattice the rotation \( Y = aXa' \) gives \( p, -q, -r \), as an equivalent point. By subtraction we find that \( 2p, 0, 0 \) and \( 0, 2q, 2r \) must be points of the lattice. But by the choice of the reference set \( 1, 0, 0 \) and \( 0, m, n \) are points of the lattice. \( p, q, r \), therefore must have the form \( \frac{1}{2}, \frac{m}{2}, \frac{n}{2} \). When \( m \) and \( n \) are both odd, \( m \) and \( n \) cannot both be even. Now as \( b \) and \( c \) were chosen arbitrarily in the first place we can always choose them so that either \( 1, m, n \) are always even or so that \( m, n \) is always even and \( 1 \) and \( m \) both even or both odd. Thus there are two types of lattice whose formulae may be written.

\[
\Gamma_{21} = \{ la + mb + nc \}
\]

\[
\Gamma_{22} = \{ \frac{(1+m)a + (1-m)b}{2} + nc \}
\]
a, b, c are here chosen along the three diad axes. Writing the formula of the lattice in the first place as \( la + mb + nc \), we can show exactly as in the previous case that the points \( 2pa, 2qb, 2rc \) belong to the lattice \( pqr \), must have the form \( \frac{1}{2}, \frac{m}{2}, \frac{n}{2} \). Here however as \( a, b, c \) are geometrically indistinguishable, there are four types \( \Gamma_0, \Gamma'_0, \Gamma''_0, \Gamma'''_0 \) according as \( \Gamma_0, 1, m, n \) are always even; \( \Gamma'_0 \), 1 (say) is always even and the other is even or they are all even; \( \Gamma''_0 \), two of the numbers are odd and \( m \) and \( n \) are both odd or both even; \( \Gamma'''_0 \), all odd or all even. The formulae of these are written

- **31** \( \Gamma_0 \) 
  \[ \frac{1}{2}a + mb + nc \]
  With the conditions \( S_{bc} = S_{ca} = S_{ab} = 0 \)

- **32** \( \Gamma'_0 \) 
  \[ \frac{1}{2}a + \frac{m^2 + nb + m - n}{2} \]

- **33** \( \Gamma''_0 \) 
  \[ -\frac{1 + m + n}{2} a + \frac{1 - m + nb}{2} + \frac{1 + m - n}{2} \]

- **34** \( \Gamma'''_0 \) 
  \[ \frac{m + n}{2} a + \frac{m + nb + 1 + m - n}{2} \]

**4** \( D_{4h} \) a is chosen along the tetrad axis b and c along diad axes then either b + c are primitives or \( (b + c) \) and \( (b - c) \) are, but as these are also diad axes the same \( c \) can always be chosen as primitives. Since a is also a diad axis this is only a special case of \( Q_4 \) but here there can be no types corresponding to \( \Gamma'_0 \) and \( \Gamma''_0 \) because when 1 is even \( m \) and \( n \) must also be even, so that there are only two types: \( \Gamma_0 \), 1, m, n always even; \( \Gamma'_0 \), 1, m, n all odd or all even. Their formulae are written as before

- **41** \( \Gamma_0 \) 
  \[ \frac{1}{2}a + mb + nc \]  with the conditions \( S_{ab} = S_{ac} = S_{bc} = 0 \)

- **42** \( \Gamma'_0 \) 
  \[ -\frac{1 - m - na}{2} + \frac{1 - m + nb}{2} + \frac{1 + m - n}{2} \]  \( b = c' \)
\( \alpha, \beta, \gamma \) are chosen along the three tetrad axes, and since all three are diad axes as well, the case is similar to that of \( \alpha \) except that in this case symmetry makes \( T' \) the same as \( T'' \) so that in this case there are three types only

\[
T' = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c
\]

With the conditions

\[
\begin{align*}
S_{bc} &= S, S_{ca} = S, S_{ab} = 0 \\
\alpha &= b = c
\end{align*}
\]

In this and the succeeding class four reference vectors will be chosen, this is not strictly necessary but it exhibits the symmetry of the lattices. Whenever four vectors are taken the linear relation between them of the type \( pA + qB + rC + sD = 0 \) imposes a relation of the type \( tp + uq + vr + ws = 0 \) upon the coordinates of any point. If the coordinates satisfy such a relation there can be no ambiguity about them.

\( a \) is taken along the triad axis, \( b, c, d \) along the diad axes perpendicular to it. The formulae of the lattices can then be written as

\[
\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d
\]

with the condition \( q + r + s = 0 \). In the net \( /qb + rc + sd \) any pair of \( b, c, d \) are primitives. Any point of the net is \( mb + nc \) but to satisfy the condition we must add \( t(a + b + c) = 0 \) so that \( m + n + t + t = 0 \), \( 3t \) must be an integer. The most symmetrical expression that satisfies these conditions for points in the net is

\[
\frac{21 - m - n}{3} + \frac{m - n}{3} - c + \frac{-1 - m + 2n}{3}
\]

with the condition \( m + n = 0 \). Now by the rotation about \( a \) there are the three corresponding points of the lattice \( p, q, r, s; p, q, r, s \) and \( p, r, s, q \) hence we must have the point \( 3p, (q + s + r), (r + q + s), (s + r + q) \) that is the point \( 3p, 0, 0, 0 \). Therefore \( 3p = 1 \) as \( t = \frac{1}{3} \).

Also by rotation about \( b \) there are the corresponding points
and since these must be points of the net \((2l-m-n)b/3 + q,r,s\) must be of the form \(1/3, m/3, n/3\). Now if since \(p+q+r+s=0\) 
\((2m-n-o)/3, (-m+2n-o)/3, (-m-n+2o)/3\)
we may write \(q,r,s\) either as \((-l, -m, -n)/3, (-l, -m, -n)/3, (-l, -m, -n)/3\);
This leads to two lattices: \(\Gamma^1_h\) written

\[1a+(2m-n-o)b/3+(-m+2n-o)c/3+(-m-n+2o)d/3\]

For \(l\) must be an integer as the remainder of the expression is the
formula of the net \(l=0\); and \(\Gamma^1_h\) written

\[(l+m+n)a/3+(m-n)b/3+(n-l)c/3+(l-m)d/3\]

\(D^6_h\) Every lattice with a hexad axis also has a triad one so that
the only type of lattice that can correspond to \(D^6_h\) must be one of
those corresponding to \(D^4_h\) namely \(\Gamma^4_h\) and \(\Gamma^4_h\). Now in the class \(D^6_h\)
there are corresponding points \(p,q,r,s\) and \(-p,q,r,s\); and hence the
point \(2p,0,0,0\). \(p\) must be of the form \(1/2\) but \(p\) cannot at the
same time be of the form \(1/3\) unless \(p\) is an integer, and this is
only the case for \(\Gamma^4_h\) so that there is only one type of hexad latti

This completes the number of possible lattices. There are
fourteen in all and every lattice can be reduced to one of these.
CHAPTER V.

GENERAL SYMMETRY RELATIONS

.0 Possible forms of the General Symmetry relation.

Returning to the general symmetry relation which we may write

$$Y = (-1)^k \frac{1}{A^a} \frac{1}{X^a} + C$$

where $k = 2, 3, 4$ or 6

we are now in a position to examine the possible values of the constant vector $C$ and their significance.

.1 Screw and Rotation Axes.

Consider the most general relation of the first sort written in the form

$$Y = a^a ax + ua + vb + wc$$

where $a$, $b$, $c$ are primitive vectors of the skeletal lattice. If the origin is on the axis $a$ it can only be related to the points $pa$ and consequently $v = w = 0$ and the relation takes the form

$$Y = a^a ax + ua$$

Repeating this $k$ times we obtain

$$Y = X + kua$$

$u$ has one of the values $\frac{n}{k}$; but $u$ must also have the value $\frac{m + \frac{n}{k}}{2k}$ so that the value of $n$ may always be made to lie between such limits as

(i) $0 < n < k$ or (ii) $\frac{-k}{2} < 0 < \frac{k}{2}$

As it is always possible to choose one of the primitive vectors along any axis of the system the same argument applies to all axes, and in general if $k$ so that we may write the general relation of the first sort in the form

$$Y = (pa + qb + rc)^a \frac{1}{X} (pa + qb + rc)^b + \frac{1}{X} (pa + qb + rc) + ua + vb + wc$$

where $a$, $b$, $c$ are the reference vectors of one of the 14 lattices $h$ is an integer and $S (pa + qb + rc)(ua + vb + wc) = 0$
When \( h \) is not 0 such a relation is called a screw; if \( h = 0 \) it is called as before a rotation. Consider the two screws
\[
Y = a^h x a^{-h} + \frac{ha}{k} \quad \text{and} \quad Y = a^h x a^{-h} - \frac{ha}{k}
\]
where \( h < \frac{k}{2} \). The second may be written in the inverse form \( Y = a^{-h} x a^{h} + \frac{ha}{k} \) which shows that it differs from the first only in the sense of its rotation, if the sense of one is that a right handed screw the second is left handed, and vice versa.

If however \( k = 2 \) or \( 2h = k \) right and left handed screws are indistinguishable.

When \( h \) is a factor of \( k \) the screw reduces to
\[
Y = a^h x a^{-h} + \frac{ha}{k'} \quad \text{where} \quad kh = k
\]
Repeating \( k \) times we have \( Y = a^h x a^{-h} + a \), there is a rotation \( Y = a^h x a^{-h} \).

The possible types of screws and rotations are given in the table.

They are expressed as if the axis passed through the origin and with a primitive vector \( a \) as axis but this is only for convenience sake.
### Types of screws and rotations.

**General form** 
\[ Y = a \cdot x^k + h \]

**Particular forms**

<table>
<thead>
<tr>
<th>( k = 2 )</th>
<th>( h = 0 )</th>
<th>( Y = a \cdot x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Y = a \cdot x^2 + a/2 )</td>
<td>or ( Y = a \cdot x^2 - a/2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( k = 3 )</th>
<th>( h = 0 )</th>
<th>( Y = a \cdot x^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Y = a \cdot x^3 + a/3 )</td>
<td>or ( Y = a \cdot x^3 - a/3 )</td>
</tr>
<tr>
<td>2</td>
<td>( Y = a \cdot x^3 + 2a/3 )</td>
<td>or ( Y = a \cdot x^3 + a/3 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( k = 4 )</th>
<th>( h = 0 )</th>
<th>( Y = a \cdot x^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Y = a \cdot x^4 + a/4 )</td>
<td>or ( Y = a \cdot x^4 - a/2 )</td>
</tr>
<tr>
<td>2</td>
<td>( Y = a \cdot x^4 + a/2 )</td>
<td>or ( Y = a \cdot x^4 + 3a/4 )</td>
</tr>
<tr>
<td>3</td>
<td>( Y = a \cdot x^4 + a/4 )</td>
<td>or ( Y = a \cdot x^4 + a/4 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( k = 6 )</th>
<th>( h = 0 )</th>
<th>( Y = a \cdot x^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Y = a \cdot x^6 + a/6 )</td>
<td>or ( Y = a \cdot x^6 + a/6 )</td>
</tr>
<tr>
<td>2</td>
<td>( Y = a \cdot x^6 + a/3 )</td>
<td>or ( Y = a \cdot x^6 + a/3 )</td>
</tr>
<tr>
<td>3</td>
<td>( Y = a \cdot x^6 + a/2 )</td>
<td>or ( Y = a \cdot x^6 + 5a/6 )</td>
</tr>
<tr>
<td>4</td>
<td>( Y = a \cdot x^6 + 2a/3 )</td>
<td>or ( Y = a \cdot x^6 + a/6 )</td>
</tr>
<tr>
<td>5</td>
<td>( Y = a \cdot x^6 + 5a/6 )</td>
<td>or ( Y = a \cdot x^6 + a/6 )</td>
</tr>
</tbody>
</table>

Including the diad axis \( Y = a \cdot x^2 \)

Including the diad axis \( Y = a \cdot x^4 \)

Including the triad \( Y = a \cdot x^6 \)
Axes not through the origin

We may now write the general relation of symmetry of the first sort in the form \( Y = A^{-1} X A^T + pA + B \) where \( S.AB = 0 \)

Let us change the origin to a point \( C \) where \( S \cdot AC = 0 \). If \( X \) and \( Y \) are the new values of \( X \) and \( Y \) then \( X = X' + C \); \( Y = Y' + C \) and the relation becomes

\[
Y' + C = A^{-1} (X + C) A + pA + B
\]

\[
Y = A^{-1} X A + pA + CA - C + B
\]

The axis passes through the new origin if

\[
A^{-1} CA - C + B = 0
\]

This equation gives the position of any screw or rotation axis in terms of the constants of its symmetry relation.

An important particular case is when \( A \) is the reference vector \( a \), and \( B \) takes the form \( vb + wc \). \( S ' b a = S . 0 0 = 0 \)

If also \( a^T b = \beta + g \), \( a^T c = \alpha + g \), the equation for a point on the axis takes the form \( x(f, b + g c) + y(f, b + g c) - xb + yc = 0 \)

which gives for \( x \) and \( y \)

\[
(x - f, y) = (1 - f, x - f, y) = w
\]

\[
-g, x + (1 - g, y) = w
\]

Relations of the second sort

As before we may write the general relation of the second sort in the form

\[
Y = A^{-1} X A + pA + B \quad \text{where} \quad S.AB = 0
\]

Transferring the origin to the point \( \frac{1}{2} B + B' \). \( S.AB' = 0 \)

we have

\[
Y = A^{-1} X A + -A - B A B + B
\]

And if \( B' \) is chosen to satisfy \( \frac{1}{2} B' A^{-1} B' = 0 \) this reduces to

\[
Y = -A^{-1} X A - \frac{1}{2}
\]

Thus the general relation of the second sort, except in the particular case \( k = 2 \) whose centre is in general not the origin. In particular the general inversion \( Y = -X + B \) is a centre of symmetry at the point \( B \).
Reflections and glide planes

In the case \( k = 2 \) the equation for the centre becomes

\[
\begin{align*}
\mathbf{A}\mathbf{A'} + \mathbf{B'} - \mathbf{B} &= 0 \\
-\mathbf{B} + \mathbf{B'} &= 0
\end{align*}
\]

From this equation we see that \( \mathbf{B} = \mathbf{A'} \mathbf{A} \). And that in general the equation cannot be reduced any further than

\[
\mathbf{Y} = -\mathbf{A}\mathbf{A'} + \mathbf{B}
\]

This relation repeated once gives

\[
\mathbf{Y} = \mathbf{X} + 2\mathbf{B}
\]

\( \mathbf{B} \) must take the form \( \frac{1}{2}(p\mathbf{a} + q\mathbf{b} + r\mathbf{c}) \) where \( p\mathbf{a} + q\mathbf{b} + r\mathbf{c} \) is a vector of the skeletal lattice, this may always be reduced to one of the forms

\( 0, 0, 0; \frac{1}{2}, 0, 0; \frac{1}{2}, \frac{1}{2}, 0; \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \); or if \( p, q, r \) are of the form

\[
\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \text{ or } \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \text{ to } \frac{1}{2}, 0, 0 \text{ or } \frac{1}{2}, 0, 0 \text{ etc}
\]

If the \( \mathbf{plane} \) can be reduced to the form \( \mathbf{Y} = -\mathbf{A}\mathbf{A'} + p\mathbf{A} \) where \( S \mathbf{A}\mathbf{B} = 0 \),

it is called a plane reflection; if it reduces to the form

\[
\mathbf{Y} = -\mathbf{A}\mathbf{A'} + p\mathbf{A} + \mathbf{B}
\]

it is called a gliding reflection; in either case the plane passes through the point \( \frac{1}{2}p\mathbf{A} \).

Parameters of the general symmetry relation.

If the general relation be written in the form

\[
\mathbf{Y} = (-1)^{\epsilon} \mathbf{A'} \mathbf{A}^{-1} + u\mathbf{a} + v\mathbf{b} + w\mathbf{c}
\]

\( u, v, w \) are called the parameters of the relation, they fix the type of the relation when the axis and rotation are given. The origin can always be chosen either to be on the axis of rotation or in the plane of rotation, which allows \( u, v, w \) to be given any desired values subject to certain restrictions; one for an axis, two for a plane, none for a centre of symmetry.
The symmetry of a point system is completely determined when the
rotations and parameters of its symmetry relations are known
\( y = \frac{1}{k} \sum_k x_k y_k + u_k y_k + w_k \) are known
as chosen, the same origin being taken for all of them. A change of origin
will give different values for the parameters but there are some invariant
relations between them. Two particular cases are of special importance.

41 If three of the relations are of the form

\[
\begin{align*}
Y &= a^k X_k + u_k y_k + w_k & \text{Where} & S \cdot bc = S \cdot ac = S \cdot ab = 0 \\
Y &= b^k X_k + u_k y_k + w_k & \text{or} \\
Y &= c^k X_k + u_k y_k + w_k \\
\end{align*}
\]

Here it is easily seen that change of origin does not alter the
values of the three quantities \( u_x, v_x, w_x \); and that any three
parameters, one chosen from each of the pairs \( u_x, u_y, u_z \); \( v_x, v_y, v_z \); \( w_x, w_y, w_z \) can be
given any desired value.

42 If the class of symmetry is hemimorphic and it has relations

\[
\begin{align*}
Y &= a^k X_k + u_k y_k + w_k & \text{Where} & S \cdot ab = S \cdot ac = 0 \\
Y &= b X_k + u_k y_k + w_k \\
Y &= c X_k + u_k y_k + w_k \\
\end{align*}
\]

Here only \( v_x, v_y, v_z \) and \( w_x, w_y, w_z \) are unaltered by change of origin and only two
of \( v_x, v_y; w_x, w_y \) can be given any desired value.

5 Relations between parameters

In general the set of parameters in any class of symmetry
are not independent, they are connected by relations and can only
be determined when some of them; usually one, two or three are known.
As these can only have some few fractional values the number of
types of symmetry in any class is always finite. It is the purpose
of the next section to derive all these types but it is first necessary
to obtain in their most general form some of the relations connecting the parameters of different symmetry relations.

**Relations with the same rotation**

**Parallel axes**

Consider the symmetry relations
\[
Y = A'X + hA/k + B \quad \text{where} \quad S_{AB} = S_{AB'} = 0
\]
\[
Y = A'X + hA'/k + B'
\]
Subtracting we have \( Y - Y = (h' - h)A + B - B' \), \( h' - h \) A + B - B' must be a point of the skeletal lattice. In general \( A = pa + qb + rc \), \( B = ua + vb + wc \), \( B' = ua' + vb' + wc' \) with the conditions
\[
\begin{align*}
\zeta u \alpha + \zeta (\nu r + wq) S_{bc} &= 0 \\
\zeta u \alpha' + \zeta (\nu' r' + w' q') S_{bc} &= 0
\end{align*}
\]
then if \( pa + qb + rc \) represents a vector of the lattice, we have
\[
\frac{K - h}{K} (pa + qb + rc) + (u' - u) a + (v' - v) b + (w' - w) c = pa + qb + rc
\]
which gives the three conditions
\[
\begin{align*}
\frac{K - h}{K} p' & = p \\
\frac{K - h}{K} q' & = q \\
\frac{K - h}{K} r' & = r
\end{align*}
\]
An important particular case \( R = 1; \nu' = 0; \nu r = 0; S_{ab} = S_{ac} = 0 \)
\( u' = u = 0 \)
\[
\frac{K - h}{K} = p; \quad v' = q; \quad w' = r
\]
If \( p, q, r \) take the form \( 1, m, n \) \( h' - h = kl \); all parallel screw axes must have the same pitch.

**Parallel planes**

Consider the planes
\[
Y = aX + ua + vb + wc \quad ; \quad Y' = a'X + u'a + v'b + w'c.
\]
Subtracting and equating to a vector of the lattice, we obtain
\[
\begin{align*}
u' - u &= p \\
v' - v &= q \\
w' - w &= r
\end{align*}
\]
In the particular case where \( A = a \) and \( p, q, r \) are \( 1, m, n \) we see that parallel planes are all reflection or all glide planes and that they are placed at distances \( \frac{1}{2}l \) apart.
62. Centre of symmetry

Consider the two centres \( Y = \mathbf{-X} + \mathbf{ua} + \mathbf{vb} + \mathbf{wc} ; Y = \mathbf{-X} + \mathbf{u}a + \mathbf{vb} + \mathbf{wc} \)
subtracting and equating to a vector of the lattice we obtain
\[ u' - u = p ; v' - v = q ; w' - w = r \]
there are centres of symmetry at all the points
\[ \frac{1}{2}(u+p)a + \frac{1}{2}(v+q)b + \frac{1}{2}(w+r)c \]

7. Relations with different rotations

71. Perpendicular axes

Consider the relations
\[ Y = a'Xa + ha/k \quad \text{where} \quad Sabc - S.ac = 0 \]
\[ Y = (qb + ro)x(qb + ro)' + ua + vb + wc \]
Transforming the first by the second we obtain
\[ Y = a'^2Xa + ha/k - a'^2(vb + wc)a + vb + wc. \]
subtracting the original relation and equating to a point of the lattice
\[ a'^2(vb + wc)a - vb - wc = pa + qb + rc \]
And if \( a'^2ha = fb + gc \); \( a'oa' = f_k + gc \) we have
\[ \frac{v(f-k-1) + wf_k}{v(g_2 - 1)^f} = q \]
\[ \frac{v(f-k-1) + wf_k}{v(g_2 - 1)^g} = r \]

72. Axis and parallel plane

Consider the relations
\[ Y = a'^2Xa + ha/k \]
\[ Y = -(qb + ro)x(qb + ro)' + ua + vb + wc \]
transforming the first by the second we obtain
\[ Y = a'^2Xa - ha/k - a'^2(vb + wc)a + vb + wc \]
This is seen to be an screw of opposite pitch. Subtracting from the original relation and equating we have:
\[ 2ha/k + a'^2(vb + wc)a - vb - wc = pa + qb + rc \]
If as before \( \frac{4}{5} a' B + \frac{2}{5} b' + c' \); \( \frac{3}{5} a' B' + c' \); we have

\[ 2h = kp; \quad \frac{v}{2} (g-1) + w' = q; \quad \frac{v}{2} + w' (g-1) = r. \]

73 **Axis and Centre of Symmetry.**

Consider the relations

\[ Y = a X + \frac{2}{5} + b + c \quad \text{where} \quad S_{ab} = S_{ac} = 0 \]

\[ Y = X + \frac{a}{2} + b + c \]

transforming the first by the second we obtain

\[ Y = a X + - b + c + \frac{2}{5} (b + c) a + \frac{2}{5} (b + c) \]

a screw of opposite pitch. Subtracting and equating as before

\[ 2h = k' + (2g - 1) + \frac{2}{5} (b + c) + \frac{2}{5} (b + c) \]

as above this gives

\[ 2h = kp; \quad 2g + (g-1) + \frac{2}{5} = q; \quad 2w + g + (g-1) + \frac{2}{5} = r \]

In this and in the previous case if \( p \) is of the form \( 1 \), \( h \) is of the form \( \frac{1}{k} \).
6.0 We are now in a position to derive the possible types of symmetry of point systems in three dimensions. The plan adopted in the work is as follows.

The symmetry relations of every class are given in their general form. Where, however, any class is derived from another by the addition of symmetry relations, only these relations are given. The conditions connecting the parameters are then found, the origin is chosen and parameters the possible reduced values of such as are chosen to be independent determined. Corresponding to each permutation of these there is a type of symmetry but some of these types may be shown to be geometrically indistinguishable, and will not count as separate types. In general there are several lattices which include any class of symmetry. The types of symmetry corresponding to these lattices will be derived in turn. The lattice whose formula may be written /la+mb+nc where a, b, c are axes of symmetry is dealt with first, to every type of symmetry which it gives rise to, there corresponds one for the other lattices of the same class, but here they are not all distinct.

The types are designated by their Schoenflies symbols even when the order of their derivation would suggest different notation.

In some cases the position and nature of axes and planes are given but these can always be obtained by from the parameters. The formulae of the various types of symmetry are reserved for a later chapter.
.1 Triclinic Types

Two classes C, and C, one lattice T.

.11 C, only the identity relation \( Y = X + uA + vB + wC \) \( u, v, w \) always reducible to \( 0, 0, 0 \). only one type C.

.12 C, only the centre \( Y = -X + uA + vB + wC \) \( u, v, w \) always reducible to \( 0, 0, 0 \). only one type C.

.2 Monoclinic Types

Three classes C, C, C. Two lattices T and T.

.21 C, the axis \( Y = xA + uA + vB + wC \) The origin can always be chosen to make \( v = w = 0 \). By 5.1 \( u = 0 \) or \( \frac{1}{2} \) 591 shows that for \( T \) \( u = 0 \) and \( u = \frac{1}{2} \) give distinct types but that for \( T' \) \( u = 0 \) implies \( u = \frac{1}{2} \) so that there is only one type. In all there are the three types

- C, skeletal lattice \( T \), parameters \( 0, 0, 0 \)
- C, " " \( T \) " \( \frac{1}{2}, 0, 0 \)
- C, " " \( T' \) " \( 0, 0, 0, \) and \( \frac{1}{2}, 0, 0 \).

.22 C, The plane \( Y = xA + uA + vB + wC \) The origin can always be chosen to make \( x = 0 \). \( v \) and \( w \) can have the values \( 0 \) or \( \frac{1}{2} \). In band \( c \) have been chosen arbitrarily so that the values for \( v, w \) \( \frac{1}{2}, 0 ; 0, \frac{1}{2} ; \frac{1}{2}, \frac{1}{2} \) are geometrically indistinguishable so that there are only two distinct types. In by 562 the values \( 0, 0 \) imply \( \frac{3}{2}, 0 \) but there are still two types since \( b \) and \( c \) are no longer arbitrary, and the values \( 0, \frac{1}{2} ; \frac{1}{2}, \frac{1}{2} \) are distinct from the former pair. In all there are the four types

- C, skeletal lattice \( T \), parameters \( 0, 0, 0 \)
- C, " " \( T \) " \( 0, \frac{1}{2}, 0 \)
- C, " " \( T' \) " \( 0, 0, 0 \) and \( 0, \frac{3}{2}, 0 \)
- C, " " \( T' \) " \( 0, 0, \frac{1}{2} \) and \( 0, \frac{1}{2}, \frac{1}{2} \)
.23 $C_{24h}$ Derived from $C_2$ by the addition of the centre

$Y = -x + u + vb + wc$

The origin can still be chosen so that

$\sigma_h$ we have for $\Gamma_{m'0}$

$2\sigma = 1; 2w = m$; and for $\Gamma_{m'0}^'$

$u = \frac{1-m}{2}; 2v = \frac{1-m}{2}; 2w = n$

but since $u$ cannot $= \frac{1}{2}$ these both give the same possible values for the

parameters namely $0, 0, 0; 0, \frac{1}{2}, 0; 0, 0, \frac{1}{2}; 0, \frac{1}{2}, \frac{1}{2}$. For $\Gamma_{m'}$

the last three are geometrically indistinguishable, for $\Gamma_{m'}^'$ the first two and the last

two imply each other. In all there are six types

$C_{24h}^'$ skeletal lattice $\Gamma_{m'}^'$, derived from $C_2^'$, parameters $0, 0, 0$

$C_{24h}^'$

$u = \Gamma_{m'}^'$

$C_{24h}^f$

$u = \Gamma_{m'}^f$

$C_{24h}^g$

$u = \Gamma_{m'}^g$

$C_{24h}^h$

$u = \Gamma_{m'}^h$

In all there are six types

$.3$ Orthorhombic Types.

Three classes $Q, Q, Q, Q$; four lattices

$.31 Q$ Three axes $Y = ax + u + vb + wc$

$Y = bX + u + vb + wc$

$Y = cZ + u + vb + wc$

Combining the three we obtain $Y = abXY + u + (v + u) + (w + u) + (w + v) + (w + v) + (w + v)$

and since $Uabc = -1$ we have

$u + u = \frac{1}{2}, \frac{1}{2}$

$-v + w + v = \frac{1}{2}$

$w = \frac{1}{2}$

we can always choose the origin to make $u + v + w = 0$

$u, v, w$, can take the values 0 or $\frac{1}{2}$; here $a, b, c$ are geometrically

indistinguishable this leads to four types for $\Gamma_{m'}$, for $\Gamma_{m'}^'$ these reduce
to two since the values 0, 0, 0 for \( u, v, w \), implies the values 0, \( \frac{1}{2}, \frac{1}{2} \)
and the values \( \frac{1}{2}, 0, 0 \) implies \( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \); for \( \Gamma' \) all the types reduce to
one; for \( \Gamma'' \) there are again two types 0, 0, 0 implying \( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \), and \( \frac{1}{2}, 0, 0 \)
\( 0, \frac{1}{2}, \frac{1}{2} \), so that there are altogether nine types.

<table>
<thead>
<tr>
<th>Q' skeletal lattice</th>
<th>( \Gamma' )</th>
<th>parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q^1 = r )</td>
<td>( \Gamma' )</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0, 0, 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \frac{1}{2}, 0, 0 )</td>
</tr>
<tr>
<td>( q^2 = r )</td>
<td>( \Gamma' )</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \frac{1}{2}, 0, 0 )</td>
</tr>
<tr>
<td>( q^3 = r )</td>
<td>( \Gamma' )</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \frac{1}{2}, 0, 0 )</td>
</tr>
<tr>
<td>( q^4 = r )</td>
<td>( \Gamma' )</td>
<td>0, 0, 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \frac{1}{2}, 0, 0 )</td>
</tr>
</tbody>
</table>

\( q^5 = r' \)

\( q^6 = r' \)

\( q^7 = r'' \)

\( q^8 = r'' \)

\( q^9 = r''' \)

\( q^{10} = r''' \)

\( q^{11} = r''' \)

\( q^{12} = r''' \)

\( q^{13} = r'''' \)

\( q^{14} = r'''' \)

\( q^{15} = r'''' \)

\( q^{16} = r'''' \)

\( q^{17} = r'''' \)

\( q^{18} = r'''' \)

\( q^{19} = r'''' \)

From these parameters we can see that
- \( Q' \) has three intersecting diad axes; \( \tilde{Q} \) has one diad screw meeting both
- of two non intersecting diad axes; \( \tilde{Q} \) has one diad axis passing through
- the centres of the squares formed by two intersecting diad screws;
- \( \tilde{Q} \) has three non intersecting diad screws; \( \tilde{Q} \) is a combination of \( \tilde{Q} \) and \( \tilde{Q} \).
$Q'$ of $Q'$ and $Q'$; $Q'$ combines all the types $Q', Q', Q', Q'$; $Q'$ combines $Q'$ and $Q'$, $Q'$, $Q'$ and $Q'$.

32. An axis and two planes

\[ Y = aX + \eta + \eta \]
\[ Y = bX + \xi + \xi \]
\[ Y = cX + \zeta + \zeta \]

Combining the three we obtain $Y = ab\lambda \eta + (u + u_1 + u_3)\lambda + (v - v_1 - v_3)b + (w - w_1 - w_3)c$ and since $Uabc = -1$ we have; for $\Gamma$, $\Gamma'$, $\Gamma''$, $\Gamma'''$, $\Gamma''''$,

\[
\begin{array}{cccccc}
\text{u} + \text{u}_1 + \text{u}_3 & L & 1 & \frac{m+n}{2} & \frac{-1+m4n}{2} & \frac{i+1+n}{2} \\
\text{v} - \text{v}_1 - \text{v}_3 & m & \frac{m+n}{2} & \frac{n+1}{2} & \frac{1-m+n}{2} & m \\
w - w_1 - w_3 & n & \frac{m-n}{2} & \frac{1+m}{2} & \frac{1-m-n}{2} & \frac{1-n}{2}
\end{array}
\]

we can always choose the origin to make $w = w_1 = 0$

$u, u_1, u_3, v, v_1, v_3$ can take the values 0 or $\frac{1}{2}$. For $\Gamma$, permutations of these lead to 16 types but since $b$ and $c$ are indistinguishable 6 pairs of these types are likewise, leaving only 10 distinct types. For $\Gamma'$ as $ab$ and $c$ are no longer indistinguishable there are two divisions according as we write $\Gamma'$ as $1a + \frac{m+n}{2}b + \frac{m-n}{2}c$ or as $\frac{1+n}{2}a + mb + \frac{1-n}{2}c$. In each of these divisions several sets of types corresponding to the 10 above imply each other which reduces the number to 3 in the first and 4 in the second division. For $\Gamma''$, the number is reduced to two and for $\Gamma''''$ to three types. In all there are 22 types which are set forth with their parameters in the following table.
\[ C_{i\alpha}' \text{ lattice } \mathcal{L}_{\gamma}' \text{ parameters} \]

\begin{array}{cccc}
\hline
\text{Parameter} & a & b & c \\
\hline
C_{2\gamma}' & 0 & 0 & 0 \\
C_{3\gamma}' & 0 & 0 & 0 \\
C_{4\gamma}' & 0 & 0 & 0 \\
C_{5\gamma}' & 0 & 0 & 0 \\
C_{6\gamma}' & 0 & 0 & 0 \\
C_{7\gamma}' & 0 & 0 & 0 \\
\hline
\end{array}

\[ C_{i\alpha}' = T_{\gamma}' \]

\begin{array}{cccc}
\hline
\text{Parameter} & a & b & c \\
\hline
C_{2\gamma}'' & 0 & 0 & 0 \text{ and } 0 & 0 & 0 \\
C_{3\gamma}'' & 0 & 0 & 0 \text{ and } 0 & 0 & 0 \\
C_{4\gamma}'' & 0 & 0 & 0 \text{ and } 0 & 0 & 0 \\
C_{5\gamma}'' & 0 & 0 & 0 \text{ and } 0 & 0 & 0 \\
C_{6\gamma}'' & 0 & 0 & 0 \text{ and } 0 & 0 & 0 \\
C_{7\gamma}'' & 0 & 0 & 0 \text{ and } 0 & 0 & 0 \\
\hline
\end{array}
<table>
<thead>
<tr>
<th>Lattice Parameters</th>
<th>( c'_{2v} )</th>
<th>( c'_{4v} )</th>
<th>( c'_{4v} )</th>
<th>( c'_{4v} )</th>
<th>( c'_{4v} )</th>
<th>( c'_{4v} )</th>
<th>( c'_{4v} )</th>
<th>( c'_{4v} )</th>
<th>( c'_{4v} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
<tr>
<td>( \Gamma'^{rr} )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
<td>( 0 \ 0 \ 0 )</td>
</tr>
</tbody>
</table>
.33 $Q'_A$ Derived from $Q$ by the addition of the centre of symmetry

$$Y = -x + u + v + w$$

By $573$, we have the possible values of $u, v, w$ are $0, 0, 0; \frac{1}{2}, 0, 0; 0, \frac{1}{2}, 0; 0, 0, \frac{1}{2}$; except in types derived from $Q'$,

$$0, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 0, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, 0; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$$. For $Q'$ since $a, b, c$ are equivalent this reduces to four types. For $Q'$, $0, 0, 0$ gives rise to indistinguishable types leaving four.

For $\frac{1}{2}, 0, 0$ & $0, 0, 0$ & $\frac{1}{2}, \frac{1}{2}, 0$ & $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ give rise to indistinguishable types leaving four.

For $\frac{1}{2}, \frac{1}{2}, 0 & 0, 0, \frac{1}{2}$ and $\frac{1}{2}, 0, 0 & \frac{1}{2}, \frac{1}{2}, 0$ do the same leaving six types.

For $Q', 0, 0, 0$ & $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and $\frac{1}{2}, 0, 0$ .... $0, \frac{1}{2}, \frac{1}{2}$ give rise to indistinguishable types leaving only two.

For $Q'$ and $Q'$ by $573$, $0, 0$ implies $0, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 0, 0; \frac{1}{2}, \frac{1}{2}$; $0, \frac{1}{2}, 0, 0, \frac{1}{2}$ and $\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}$ also in $Q'$, $0, 0, 0$ & $\frac{1}{2}, 0, 0$ and $\frac{1}{2}, 0, 0$ & $\frac{1}{2}, \frac{1}{2}, 0$ give rise to indistinguishable types but they do not do so in $Q'$ so that four latter types correspond to the former and only two to the former.

For $Q'$ by $573$, $0, 0$ implies $0, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 0, 0; \frac{1}{2}, \frac{1}{2}$; $0, \frac{1}{2}, 0, 0, \frac{1}{2}$ and $\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}$; $0, \frac{1}{2}, 0, 0, \frac{1}{2}$ so that there are only two types.

For $Q$ and $Q'$ by $573$, $0, 0$ implies $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 0, 0; \frac{1}{2}, \frac{1}{2}$; etc and as the last three give rise to indistinguishable types these are only two types corresponding to each of $Q$ and $Q'$.

In all there are 28 types.

<table>
<thead>
<tr>
<th>Lattice $Q'$ derived from $Q$ parameters</th>
<th>0 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q'$</td>
<td>$\frac{1}{2}, \frac{1}{2}$</td>
</tr>
<tr>
<td>$Q'$</td>
<td>0 0 0</td>
</tr>
<tr>
<td>$Q'$</td>
<td>$0, \frac{1}{2}$</td>
</tr>
<tr>
<td>$Q'$</td>
<td>0 0 0</td>
</tr>
<tr>
<td>$Q'$</td>
<td>$\frac{1}{2}, \frac{1}{2}$</td>
</tr>
<tr>
<td>$Q'$</td>
<td>0 0 0</td>
</tr>
<tr>
<td>$Q'$</td>
<td>$0, \frac{1}{2}$</td>
</tr>
</tbody>
</table>
Tetragonal Types

Seven classes $C_4, C_4', C_{4h}, D_4, D_{4h}, C_{4v}, D_{4h}$. Two lattices $\Gamma'$ and $\Gamma''$.

C. The axis $Y = a'Y + uA + vB + wC$. The origin can always be chosen to make $v = w = 0$. $u$ can have the values $0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ (the relations they represent are called tetrad axis, right, neutral, and left tetrad screws respectively). By symmetry we have for $\Gamma'$ $u' - u = 1$ for $\Gamma'' u' - u = \frac{1}{2}$ so that for $\Gamma'$ there are four classes and for $\Gamma''$ two. The position of the tetrad axes are given by $\mathbf{f}$, here $f_x = g_x = 0 \quad g_y = 1 \quad f_z = -1$ so that the coordinates of a point on the axis are given by $x + y = v \quad x - y = w$ for $\Gamma'$. This gives $x = \frac{m+n}{2} \quad y = \frac{m-n}{2}$; for $\Gamma'' x = \frac{m+n}{4} \quad y = \frac{m-n}{4}$ but here if $m$ and $n$ are both odd or even, the pitch of the axis differs by $\frac{1}{4}$ from that which passes through a point where one is odd and the other even. The tetrad axis implies the diad axis $Y = aX + 2uA$ and we have from $\mathbf{f}$ these must pass through the points $\frac{1}{2}m, \frac{1}{2}n$ for $\Gamma'$ and $\frac{m+n}{4}, \frac{m-n}{4}$ for $\Gamma''$, being all of the same kind in the first case and alternately rotation or screw axes in the second.

In all there are six types

<table>
<thead>
<tr>
<th>$C_4'$ lattice $\Gamma$ parameters</th>
<th>0 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_4'$</td>
<td>1 0 0</td>
</tr>
<tr>
<td>$C_4''$</td>
<td>1 0 0</td>
</tr>
<tr>
<td>$C_4'$</td>
<td>1 0 0</td>
</tr>
<tr>
<td>$C_4''$</td>
<td>1 0 0</td>
</tr>
<tr>
<td>$C_4'$</td>
<td>0 0 0</td>
</tr>
<tr>
<td>$C_4''$</td>
<td>0 0 0</td>
</tr>
</tbody>
</table>
42 \( C'_4 \) the axis \( Y = -a^2 x^2 + u a + v b + w c \). The origin can always be chosen to make \( u = v = w = 0 \) so that there are only two types one derived from \( \Gamma'_4 \) and one from \( \Gamma'_4 \). By subtracting \( Y = -a^2 x^2 \) we have \( u, v, w = l, m, n \) for \( \Gamma'_4 \) and \( \frac{L + m + n}{L}, \frac{1 - m + n}{L}, \frac{1 + m - n}{L} \) for \( \Gamma'_4 \). By this shows that the centres of the set of alternating tetragonal axes are at the points \( \frac{1}{2}, \frac{m + n}{L}, \frac{m - n}{L} \) in the first case and \( \frac{-1 + m + n}{2}, \frac{1}{2}, \frac{m - n}{L} \) in the second.

There are two types

\[ C'_4 \text{ lattice } \Gamma'_4 \text{ parameters } 0 \ 0 \ 0 \]

\[ C_4^* \text{ lattice } \Gamma'_4 \text{ parameters } 0 \ 0 \ 0 \text{ and } \frac{1}{2} \ \frac{1}{2} \]

43 \( C_4 \) Derived from \( C_4 \) by the addition of the centre of symmetry

\[ Y = -x^2 + u a + v b + w c \]  

The origin can be chosen to make \( u = 0 \)

Since by \( \mathcal{S} \) a centre of symmetry transforms right into left handed screws types of this class can only be derived from the types \( C'_4, C_4^*, C_4^+, C_4^6 \). we have also the conditions:

\[ 2u = 1 \]

\[ v + w = m \]

\[ v - w = n \]

\[ \frac{1 + m + n}{L}, \frac{1 + m - n}{L} \]

\[ \frac{1 - m + n}{L}, \frac{1 - m - n}{L} \]

\( v \) and \( w \) may have the values \( 0, 0 \) or \( \frac{1}{2}, \frac{1}{2} \) but for \( C'_4 \) since \( u \) is \( 0 \) or \( \frac{1}{2} \), \( v, w \) can only be \( 0, 0 \) while for \( C \) since \( u \) is \( \frac{1}{2} \) or \( \frac{1}{2} \), \( v, w \) can only be \( \frac{1}{2}, \frac{1}{2} \). There are in all six types.

\( C_4^* \) lattice \( \Gamma'_4 \) derived from \( C_4^* \) parameters 0 0 0

\[ C_4^{*a} = \Gamma'_4 \text{ " } C_4^* \text{ " } 0 \frac{1}{2} \frac{1}{2} \]

\[ C_4^{*b} = \Gamma'_4 \text{ " } C_4^* \text{ " } 0 0 0 \]

\[ C_4^{*c} = \Gamma'_4 \text{ " } C_4^* \text{ " } 0 \frac{1}{2} \frac{1}{2} \]

\[ C_4^{*d} = \Gamma'_4 \text{ " } C_4^* \text{ " } 0 0 0 \]

\[ C_4^{*e} = \Gamma'_4 \text{ " } C_4^* \text{ " } 0 \frac{1}{2} \frac{1}{2} \]

\[ C_4^{*f} = \Gamma'_4 \text{ " } C_4^* \text{ " } 0 0 0 \]
The parameters are connected by several conditions.

Transforming 2 by 1 we obtain $X = aX + (2u + v) a - wb + v$ but this may be
the same as 3 and we have $u = u + 2u; v = -w; w = v.$

Similarly for 4 and 5 $u = u - 2u; v = w; w = -v.$

Combining 1, 2, and 4 we have since $a^2 b(b+c) X(b+c)^2 = a^2 w X$

For \(\frac{T^*}{e}\)

$$u - u + u = 1 + \frac{1 + m + n}{2},$$

$$v = \frac{1 - m + n}{2},$$

$$w = \frac{1 + m - n}{2},$$

The equivalent parameters $u, v, w$ and $u, v, w + 1$ represent different axes.

By \(\frac{C^*}{e}\) we have for $T^*/e$ $v = \frac{m + n}{2}; w = \frac{m - n}{2}$ for $T^*/e$ $v = \frac{1}{2}; w = \frac{m - n}{2}$ but by \(\frac{C^*}{e}\) the values

0, 0, 0 for $u, v, w$ imply the values $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ so that these can only give

two rise to one case. There correspond **two** types of this class to each of
types the classes $C'_2, C'_4, C'_6, C'_4$, and one to the types $C'_6, C'_6$; making 10 in all.

$D'_4$ lattice $T'_4$ parameters 0 0 0 0

0 0 0

0 0 0

0 0 0

0 0 0

$D'_{4'}$ 0 0 0

$D'_{4}$ 0 0 0

$D'_{4'}$ 0 0 0

$D'_{4}$ 0 0 0

$D'_{4'}$ 0 0 0
\[
\begin{align*}
D_4^4 \text{ lattice parameters} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
D_4' \ & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
D_4'' & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
D_4''' & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

45 $D_4$ three axes and two planes

\[
Y = -x^2
\]

\[
Y = bX + ya + vb + wc \\
Y = cX + ya + vb + wc \\
Y = -(b+c)X(b+c) + ya + vb + wc \\
Y = -cX(b-c) + ya + vb + wc
\]

The relations of this class can also be written in a form where diad axes and planes change places each of these ways give rise to distinct types; it is more convenient however to consider them derived from the above relations with skeletal lattices $T_4'$, $T_4'$ written $l_a + \frac{m+b}{2} + \frac{m-b}{2}$ and $T_4''$ written $\frac{m+n}{2} + \frac{n-1}{2} + \frac{1}{2}$. 

As in $D$ we obtain the conditions:

\[
\begin{align*}
u'_4 &= -u' \\
v_4 &= v' \\
w_4 &= -w \\
u_4' &= u' \\
v_4' &= -v' \\
w_4' &= w'
\end{align*}
\]
Combining 1, 2, 3, 4 we have:

\[ u_s - u_r = 1 \quad \Gamma_r' \quad \Gamma_r'' \quad \Gamma_r^{\prime\prime}\]

\[ \frac{-1 + m + n}{2} \quad l \quad \frac{m + n}{2} \quad \frac{n + 1}{2} \]

\[ v_s + v_r = m \quad \frac{l - m + n}{2} \quad \frac{m + n}{2} \quad \frac{n + 1}{2} \]

\[ w_s - w_r = n \quad \frac{l + m - n}{2} \quad \frac{m - n}{2} \quad \frac{l + m}{2} \]

As in \( \eta \), equivalent parameters \( u, v, w \) and \( u_r, v_r, w_r \) represent different kinds of plane. The actual arrangement of axes and planes can be seen by considering the arrangement in the corresponding types of \( Q \) and \( O \).

By 5.71 we have:

\[ v_r = \frac{m + n}{2} \quad \frac{1}{2} \quad \frac{m}{2} \quad \frac{2l + m + n}{2} \]

\[ w_r = \frac{m - n}{2} \quad \frac{m - n}{2} \quad \frac{n}{2} \quad \frac{m - n}{2} \]

and by \( u_r = \frac{1}{2} \)

For \( \Gamma_r \), \( \Gamma_r'' \) since \( u, v, w \) can have the values 0, 1 there are four types each for \( \Gamma \) the values 0, 0 and \( \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2} \), and \( \frac{1}{2}, 0 \) imply each other so that there are only two, for \( \Gamma_r' \) so do 0, 0 and \( 0, \frac{1}{2}; \frac{1}{2}, 0 \) and \( \frac{1}{2}, \frac{1}{2} \) so that again there are only two.

In all there are twelve types (only the parameters \( u, v, w \) are given)

\[ u_r, v_r, w_r, \]

\[ D_{4d} \text{ lattice} \Gamma_r' \text{ corresponding to } Q' \text{ } \eta_r^{\prime\prime} \text{ parameters} \]

<table>
<thead>
<tr>
<th>( D_{4d} )</th>
<th>( \eta_r' )</th>
<th>( \eta_r'' )</th>
<th>( \eta_r^{\prime\prime} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_{4d} )</td>
<td>( u_r = 0 )</td>
<td>( v_r = 0 )</td>
<td>( w_r = 0 )</td>
</tr>
<tr>
<td>( D_{4d} )</td>
<td>( u_r = 0 )</td>
<td>( v_r = 1 )</td>
<td>( w_r = 0 )</td>
</tr>
<tr>
<td>( D_{4d} )</td>
<td>( u_r = 0 )</td>
<td>( v_r = 0 )</td>
<td>( w_r = 1 )</td>
</tr>
<tr>
<td>( D_{4d} )</td>
<td>( u_r = 1 )</td>
<td>( v_r = 0 )</td>
<td>( w_r = 0 )</td>
</tr>
<tr>
<td>( D_{4d} )</td>
<td>( u_r = 1 )</td>
<td>( v_r = 1 )</td>
<td>( w_r = 0 )</td>
</tr>
<tr>
<td>( D_{4d} )</td>
<td>( u_r = 0 )</td>
<td>( v_r = 0 )</td>
<td>( w_r = 1 )</td>
</tr>
<tr>
<td>$D_{2d}^0$ lattice corresponding to $q^0 q_2^j$ parameters</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
<td>$0$ and $\frac{1}{2}$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$D_{2d}^I$</td>
<td>$q^I q_{2r}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$D_{2d}^f$</td>
<td>$q^f q_{2r}$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$D_{2d}^r$</td>
<td>$q^r q_{2r}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$D_{2d}^m$</td>
<td>$q^m q_{2r}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$D_{2d}^l$</td>
<td>$q^l q_{2r}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$D_{2d}^n$</td>
<td>$q^n q_{2r}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
one axis and four planes

\[ Y = a^2 x^2 + u_a \\
Y = -b^2 x^2 + u_a + v_b + w_c \\
Y = -c^2 x^2 + u_a + v_b + w_c \\
Y = -\left(b + c\right)x\left(b + c\right)^{-1} + u_a + v_b + w_c \\
Y = -(b - c)x(b - c)^{-1} + u_a + v_b + w_c \]

By As in D, we have the conditions \( u = u_a, \quad v = -v_b, \quad w = v_c \)

\[ u_v = u_b, \quad v = -v_c, \quad w = -v_d \]

Combining, we have:

\[ \frac{u + u_v}{2} = -\frac{1 + m + n}{2} \]

\[ \frac{v - v_c}{2} = \frac{1 - m + n}{2} \]

\[ \frac{w + w_d}{2} = \frac{n + m - n}{2} \]

By we have

\[ \frac{u - u}{2} = \frac{1 + m - n}{2} \]

\[ \frac{v - v_c}{2} = \frac{m + n}{2} \]

\[ \frac{w - w_d}{2} = \frac{m - n}{2} \]

Now since \( u \) can only have the values 0, \( \frac{1}{2} \) and \( v, w \) can can only have 0, 0 or \( \frac{1}{2} \) there are 8 types. For \( u \) can have the values 0, \( \frac{1}{2} \), \( \frac{1}{2} \) but 0 implies \( \frac{1}{2} \) and \( \frac{1}{2} \), further, in the first case the value of \( v, w \) are limited to 0 and \( \frac{1}{2} \) and \( \frac{1}{2} \) and these imply each other; while in in the second the values are limited to \( \frac{1}{2}, 0 \) and \( 0, \frac{1}{2} \) which also imply each other, so that in each of these cases there are only two types. In all there are twelve types whose parameters are shown on the following table. (Only the \( v_w \) reduced values of the parameters \( u, v, w \), are shown the others may readily be calculated by the equations above.)
<table>
<thead>
<tr>
<th>( C_{4*} ) lattice parameters</th>
<th>0 0 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{4*} )</td>
<td>0 0 ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>0 ( \frac{1}{2} ) 0 0</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>0 ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>( \frac{1}{2} ) 0 0 0</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>( \frac{1}{2} ) 0 0 0</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>0 0 0 0 and ( \frac{1}{2} ) 0 0 0</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>0 0 0 ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>0 ( \frac{1}{2} ) 0 0</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>( \frac{1}{2} ) 0 ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} ) 0 0</td>
</tr>
<tr>
<td>( C_{4*} )</td>
<td>( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} ) ( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

\[ Y = -X + u_3 a + v_3 b + w_3 \]

As in the case of \( C_{4h} \), the only types of \( D \) from which types of \( D \) may be derived are those for which \( u = 0 \) or \( \frac{1}{2} \) or where it is \( \frac{1}{2} \) or \( \frac{1}{2} \), and at the same time namely \( D'_3, D'_4, D'_5, D'_6, D'_7, D'_8 \).

As in \( C_{4h} \), \( u, v, w \) can take the values 0 and \( \frac{1}{2} \) but since \( b \) and \( c \) are equivalent axes there are only four values 0, 0, 0; \( \frac{1}{2}, \frac{1}{2} \); \( \frac{1}{2}, 0, 0 \); \( \frac{1}{2}, \frac{1}{2} \).
Thus each of $D_{4h}^{11}$ give rise to four types. For $D_{4h}^{11}$uvw must be $0, 0, 0$ or $\frac{1}{2}, 0, 0$ and for $D_{4h}^{11}$uvw must either be $0, \frac{1}{2}, 0$ implying $0, 0, 0$ or $\frac{3}{2}, \frac{3}{2}, 0$ implying $\frac{1}{2}, 0, \frac{1}{2}$. $D_4^+$ and $D_4^-$ give rise to two types each. In all there are 20 types.

<table>
<thead>
<tr>
<th>Derived from $D_2^+$</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, 0, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, 0, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{1}{2}, 0, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, \frac{1}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{1}{2}, 0, \frac{1}{2}$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, 0, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, \frac{1}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, 0, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, \frac{1}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, 0, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, \frac{1}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, 0, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$0, \frac{1}{2}, 0$</td>
</tr>
<tr>
<td>$D_{4h}^{11}$</td>
<td>$\frac{3}{2}, \frac{3}{2}, 0$</td>
</tr>
</tbody>
</table>
Thus each of $D_{4}^{12,3}$ give rise to four types. For $D_{4}^{12,3}$ u,v,w must be 0,0,0 or $\frac{1}{2},0,0$ and for $D_{4}^{12,3}$ u,v,w must either be 0,$\frac{1}{2}$,0 implying 00$\frac{1}{2}$ or $\frac{1}{2},0,0$ implying $\frac{1}{2},0,0$. $D_{4}$ and $D_{4}^{12,3}$ give rise to two types each. In all there are 20 types.

<table>
<thead>
<tr>
<th>Derived from $D_{4}$ Parameters</th>
<th>0 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$D_{4}$</td>
</tr>
</tbody>
</table>
.5 Rhombohedral Types

Five classes $C_3, C_3', D_3, C_3, D_3'$. Two lattices $I_4, I_4'$.

.51 $C_3$ the axis

$$Y = a^3x^3 - \frac{5}{3}$$

The origin can always be chosen to make $v = w = t = 0$.

$u$ can have the values $0, 1/3, 2/3$. For $I_4'$ these are independent, the

positions of the similar triad axes or screws can be found by $5:2$ to be

$m-n : n-o : o-m$. For $I_4$ the values $0, 1/3, 2/3$ imply each other but these $\frac{41-m-n}{9}$, etc

dissimilar axes pass through the same points. There are four types

$C'_3$ lattice $I_4' u = 0$; $C''_3$ lattice $I_4' u = \frac{1}{3}$; $C''_3$ lattice $I_4' , u = \frac{2}{3}$

$C'_3$ 

$\frac{1}{3}u = 0$ and $\frac{1}{3}$ and $\frac{2}{3}$

.52 $C_{3h}$ Derived from $C_3$ by the addition of a centre of symmetry

$$Y = -x + u \alpha + v \beta + w \gamma + t$$

The origin can always be chosen to make $u = 0$.

As in the case of $C_4$ only the types $C'_3$ and $C''_3$ can give rise to a type of

this class. By $5:7$ we have: for both $I_4' I_4'$ $v = \frac{m-n}{3}$; $w = \frac{n-o}{3}$; $t = \frac{o-m}{3}$

so that there are only the two types

$C''_3$ lattice $I_4'$ derived from $C'_3$, parameters $0, 0, 0$

$C''_3$ 

$I_4'$ " $C''_3$ " $0, 0, 0$

.53 $D_3$ four axes

$$Y = a^3x^3 - \frac{5}{3}$$

the two ways of writing the relations depend on the fact that both

$b, c, d$ and $(c+d), (d+b), (b+c)$ can be taken as diaz axes of the lattice $I_4$.

though only the first can be taken for $I_4'$. 
Transforming 2 by 1 and equating to \( \mathbf{p} \), we obtain \( u_x = u + 2u; v_x = v; w_x = v; t_x = w \).

This holds for both for both forms of the relations.

By \( S \), we have \( v = \frac{m-n}{3}; w = \frac{n-0}{3}; t = \frac{0-m}{3} \), but this is of the same form as we obtained for the axial parameters in \( C_3 \), so that only one type in each form corresponds to the three types \( C_2^f, C_3^f, C_4^f \), and only one of the first forms to \( C_3^f \). There are in all seven types.

<table>
<thead>
<tr>
<th>( D_3^f ) lattice parameters</th>
<th>0 0 0 0 0</th>
<th>Diad axes</th>
<th>b, c, d</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_3^f )</td>
<td>0 0 0 0 0c</td>
<td></td>
<td>(c+d), (d+b), (b+c)</td>
</tr>
<tr>
<td>( D_3^f )</td>
<td>1 0 0 0 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_3^f )</td>
<td>1 0 0 0 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_3^f )</td>
<td>0 0 0 0 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_3^f )</td>
<td>0 0 0 0 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_3^f )</td>
<td>0 0 0 0 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_3^f )</td>
<td>0 0 0 0 3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

54 \( C_3 \) one axis and three planes

\[ Y = a_1x_1^3 + u \]

\[ Y = -b_1x_1^3 + u \]

\[ Y = -c_1x_1^3 + u \]

\[ Y = -d_1x_1^3 + u \]

Exactly as for \( D_3 \) we obtain \( u_x = u_x; v_x = v_x; w_x = v_x; t_x = w_x \).

By \( S_{2u} = 0 \) and by \( S_{2u} = 0 \) or \( \frac{1}{2} \). Further as for \( D_3 \) there can only one set of independent values of \( v, w, t \). Thus we have two types corresponding to each of \( C_3^f, C_3^f \), and to each of \( C_3^f, C_3^f \). In all six types.
\( C_3' \) lattice \( G \) parameters \( 0 \ 0 \ 0 \ 0 \ 0 \) planes perpendicular to \( b, c, d \)

\[
\begin{align*}
C_{3}' & \quad T_{3}' & \quad 0 \ 0 \ 0 \ 0 \ 0 \ & \quad n' \ & \quad n' \\
C_{3} & \quad T_{3} & \quad 0 \ 0 \ 0 \ 0 \ 0 \ & \quad n \ & \quad n \\
C_{3}' & \quad T_{4}' & \quad 0 \ 0 \ 0 \ 0 \ 0 \ & \quad \mathbf{(a+d),(d+b),(b+c)} \\
C_{3} & \quad T_{4} & \quad 0 \ 0 \ 0 \ 0 \ 0 \ & \quad \mathbf{b, c, d} \\
C_{3}' & \quad T_{5}' & \quad 0 \ 0 \ 0 \ 0 \ 0 \ & \quad \mathbf{b, c, d} \\
C_{3} & \quad T_{5} & \quad 0 \ 0 \ 0 \ 0 \ 0 \ & \quad \mathbf{b, c, d}
\end{align*}
\]

.55 \( D_3' \) derived from \( D_3 \) by the addition of a centre of symmetry

\[
Y = X + x_4 a + y_4 b + z_4 c + t_4 d
\]

Exactly as in \( C_3 \) we find that the only types of \( D_3 \) that can give rise to types of \( D_3' \) are \( D_3', D_3'^{1}, D_3'^{2} \), also that \( x_4, y_4, z_4 \), always reduce to the same values 0, 0, 0. but in this case \( x_4 \), can have the values 0 or \( \frac{1}{3} \) so that to each of \( D_3', D_3'^{1}, D_3'^{2} \), corresponds two types, making six in all.

- \( D_3' \) corresponding to \( D_3' \), \( u = 0 \)
- \( D_3'^{1} \) \( u = \frac{1}{3} \)
- \( D_3'^{2} \) \( u = 0 \)
- \( D_3'^{3} \) \( u = \frac{1}{3} \)
- \( D_3'^{4} \) \( u = 0 \)
- \( D_3'^{5} \) \( u = \frac{1}{3} \)

\[ \quad \text{and} \quad \exists \]
.6 _Hexagonal Types_

Seven classes $C_6, C_{4h}, C_{3h}, D_{3h}, D_4, C_{6v}, D_{6h}$. One lattice.

.61. $C_6$ the axis

\[ Y = a^2X + ua + vb + wc + td \]

The origin can always be chosen to make $v = w = t = 0$

\[ u \] can have the values $1/6, 1/3, 1/2, 2/3, 5/6$.

By $52$ we can find the position of the hexad axes to be $\frac{21 - m - Nb}{3}$

and as each axis is also a triad and diad axis there are such axes

trough the points $\frac{1 - M - Nb}{2}$ and $\frac{21 - m - Nb}{6}$ respectively.

There are six types $C'_6, C'_4, C_{4}^{'}, C'_3, C'_2, C'_1$. with the six values respectively, that are given above for $u$.

.62 $C_{6h}$ derived from $C_6$ by the addition of an _xxix_ of symmetry.

\[ Y = -X + ua + vb + wc + td \]

As in $C_3$, we see that the only types of $C_6$ from which types of $C_{6h}$
can be derived are $C'_6$ and $C_3'$, also that that $u, v, w, t$ always reduce to $0$.

Thus there are only two types: $G$ derived from $C_6$; $C$ derived from $C_3$.

.53 $C_{3h}$ The axis

\[ Y = a^2X + ua + vb + wc + td \]

The origin can always be chosen so that $u, v, w, t = 0, 0, 0, 0$.

There is only one type $C'_3$.

.54 $D_{3h}$ The four axes, and the three planes

\[ Y = -a^2X - ua \]

\[ Y = bXb'^{-1} + ua + vb + wc + td \] and \((c + d)X(c + d)' + etc \]

\[ Y = cXc'^{-1} + ua + vb + wc + td \]

\[ Y = dXd'^{-1} + ua + vb + wc + td \]

\[ Y = -(c + d)X(c + d)' + ua + vb + wc + td \] and \(-bXb' + etc \]

\[ Y = -(d + b)X(d + b)' + ua + vb + wc + td \]

\[ Y = -(b + c)X(b + c)' + ua + vb + wc + td \]

The relations between the parameters are similar to those in $D_3$ and $C_3$. 

Here however $u$ can be 0 or $\frac{1}{3}$ so that there two types of each sort.

$D^*_3 u = 0$ diad axes $b, c, d.$

$D^*_3 u = \frac{1}{3}$

$D^*_3 u = \frac{2}{3}$

65 $D_6$ Seven axes

$Y = a Xa + ua$

$Y = b Xb + ub + vb + wc + td$

$Y = (c + d) X(c + d) + ua + vb + wc + td$

The relations between are similar to those of $D_3$

There are six types $D^*_6, D^*_6, D^*_6, D^*_6, D^*_6, D^*_6$

$u = 0 \quad \frac{1}{6} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2}$

66 $O_6$ Six planes and one axis

$Y = a^2 Xa^2 + ua$

$Y = b Xb + ub + vb + wc + td$

$Y = (c + d) X(c + d) + ua + vb + wc + td$

The relations between the parameters are similar to those of $O_3$

$u = 0$ or $\frac{1}{2}, u = u, u = \frac{1}{6}$

67 $D_{4h}$ Derived from $D_6$ by the addition of a plane of symmetry

$Y = -X + ua + vb + wc + td$

As in the case of $O, W, V, W, T, O$, can always be reduced to 0 0 0

and $u, u, u, u$ can only have the values 0, $\frac{1}{2}$ so that again there are four types

$D^*_4 u = 0; u = 0. \quad D^*_4 u = 0; u = \frac{1}{2}. \quad D^*_4 u = \frac{1}{2}; u = 0. \quad D^*_4 u = \frac{1}{2}; u = \frac{1}{2}$
Regular Types.

Five classes $T, T_4, T_6, O, Q$. Three lattices $\Gamma, \Gamma', \Gamma''$.

1. $T$ seven axes

\[
Y = aX^1 + uA + vb + wc \\
Y = bXb' + uA + vb + wc \\
Y = cXc' + ua + vb + wc \\
Y = (a+b+c)x(a+b+c)' + ua + vb + wc \\
Y = (a-b-c)x(a-b-c)' + ua + vb + wc \\
Y = (a+b-c)x(a-b-c)' + ua + vb + wc
\]

As $\Gamma, \Gamma', \Gamma''$ may all be considered as special forms of $Q$, we must accordingly have rotation axes in the direction of each of the triad axes.

The origin may accordingly be chosen to lie on the rotation axis triad $\Delta 4b + c$ so that $u_i^* = v_j^* = w_k = 0$ also the parameters of the other rotation axes can always be made to refer to rotation axes if $u_i^* = v_j^* = w_k^*$, respectively are made $= 0$.

Transforming 1 by 4 and equating to 2, and similarly for 2 and 3, we have the conditions

\[
u_i = v_j = w_k; u_i = v_j = w_k; u_i = v_j = w_k.
\]

Now $T$ may also be regarded as a special form of $Q$ then the only types of $Q$ to which types of $T$ may correspond are those which satisfy the above conditions namely $Q', Q'', Q^q, Q^q, Q^q$ so that there five types of $T$.

Transforming 4 by 1 and equating to 5 we have

\[
\begin{align*}
4 & 2 & 2 & 6 & u_i^* = v_j^* = w_k^*; v_i^* = v_j^* = w_k^*; w_i^* = w_j^* = w_k^* \\
4 & 3 & 3 & 7 & u_i^* = v_j^* = w_k^*; v_i^* = v_j^* = w_k^*; w_i^* = w_j^* = w_k^*
\end{align*}
\]

$T'$ lattice $\Gamma'$ corresponding to $Q'$ parameters

| 0 0 0 |
| 0 0 0 |
| 0 0 0 |
| 0 0 0 |
| 0 0 0 |
| 0 0 0 |
| 0 0 0 |

$T^* \quad \Gamma' \quad Q^q$
\[ T' \text{ lattice } \Gamma' \text{ corresponding to } Q' \text{ parameters} \]

\[
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[ T' = \Gamma' \quad Q' = \]

\[
\begin{array}{cccccccccccc}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Derived from \( T \) by the addition of a centre of symmetry \( \gamma = +n + \gamma + n + \gamma \).

As in the case of \( Q \), the only possible values for \( u, v, w \) are 0 or \( \frac{1}{2} \)

but by the trigonal symmetry \( u = v = w \), so that these reduce to two

0 0 0 and \( \frac{1}{2} \frac{1}{2} \frac{1}{2} \) but just as in \( Q \) we have for \( T' \) 0 0 0 and \( \frac{1}{2} \frac{1}{2} \frac{1}{2} \)

are indistinguishable and in \( T' \) and \( T \) they imply each other so that

only \( T' \) and \( T \) give rise to two types of \( T \), thus making 7 in all.

\[ T' \text{ lattice } \Gamma' \text{ derived from } T' \text{ parameters} \]

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & & & & & & & \\
T_h' & " & \Gamma' & " & T' & " & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\
T_c' & " & \Gamma' & " & T' & " & 0 & 0 & 0 & \\
T_c' & " & \Gamma' & " & T' & " & 0 & 0 & 0 & \\
T_c' & " & \Gamma' & " & T' & " & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\
T_c' & " & \Gamma' & " & T' & " & 0 & 0 & 0 & \text{ and } \frac{1}{2} \frac{1}{2} \frac{1}{2} \\
T_c' & " & \Gamma' & " & T' & " & 0 & 0 & 0 & \frac{1}{2} \frac{1}{2} \frac{1}{2} \\
\end{array}
\]
.73 $T_d$ Derived from $T$ by the addition of the six planes

\[
Y = -(b+c)X(b+c) + ua + vb + wc
\]
\[
Y = -(b-c)X(b-c) + ua + vb + wc
\]
\[
Y = -(c+a)X(c+a) + ua + vb + wc
\]
\[
Y = -(c-a)X(c-a) + ua + vb + wc
\]
\[
Y = -(a+b)X(a+b) + ua + vb + wc
\]
\[
Y = -(a-b)X(a-b) + ua + vb + wc
\]

This class may be considered as a special case of both $D_{4d}$ and $C_{4v}$.

From the first we see that as there is no type of $D_{4d}$ corresponding to $Q'$, there can be no type of $T_d$ corresponding to $T'$. From the second we see that as in $C_{4v}$, $u \perp 0$ or $\frac{1}{2}$ so $u + v + w = 0$ or $\frac{1}{2}$ for $T_{4d}^{I}$ and $0$ or $\frac{1}{2}$ for $T_{4d}^{R}$.

Further $D_{4d}^{R}$ is the only type of $D_{4d}$ corresponding to $Q'$ and $u + v + w = 0$ also $D_{4d}^{R}$ is the only type corresponding to $T'$ and $T_{4d}^{R}$.

In all there are six types:

$T_d$ lattice corresponding to $T' D_{4d}^{I} C_{4v}^{R}$ parameters $0 \ 0 \ 0$

$T_{4d}^{I}$

$T_{4d}^{R}$

$T_{4d}^{I}$

$T_{4d}^{R}$

$T_{4d}^{I}$

$T_{4d}^{R}$

.74 $O$ Derived from $T$ by the addition of the six axes

\[
Y = (b+c)X(b+c) + ua + vb + wc
\]

\[
Y = (a-b)X(a-b) + ua + vb + wc
\]

This class may be regarded as a particular case of $D_3$ the diad and triad for axes must accordingly always intersect, also by we can see that there is always a diad axis i.e. $v + w = 0$. Also as $O$ is a particular case of $D$ we have for types corresponding to $T' T_{4d}^{I} T_{4d}^{R} T_{4d}^{I} T_{4d}^{R} T_{4d}^{I}$
There are in all then eight types. As the class 0 has tetrad axes in the place of the diad axes of $T$ the parameters $u, v, w$ are given as well as $u', v', w'.

<table>
<thead>
<tr>
<th>$O'$ lattice corresponding to $T' D'_+$ parameters</th>
<th>$O'$ lattice $\Gamma$</th>
<th>$T' D'_+$ parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$ &amp; $0$ &amp; $0$ &amp; $0$ &amp; $0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$ &amp; $\frac{1}{2}$ &amp; $0$ &amp; $0$ &amp; $0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$ &amp; $0$ &amp; $\frac{1}{2}$ &amp; $0$ &amp; $0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$ &amp; $0$ &amp; $0$ &amp; $\frac{1}{2}$ &amp; $0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$ &amp; $0$ &amp; $0$ &amp; $0$ &amp; $\frac{1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$ &amp; $0$ &amp; $0$ &amp; $0$ &amp; $0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$ &amp; $\frac{1}{2}$ &amp; $0$ &amp; $0$ &amp; $0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0$ &amp; $0$ &amp; $\frac{1}{2}$ &amp; $0$ &amp; $0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
$Y = -X + uA + vB + wC$

As in the case of $D_4$, we can only derive types of this class from classes corresponding to $D_1 D_2 D_3 D_4 D_5$ namely $0^1 O^0 0^1 O^0 0^1 O^0$.

For $\Gamma$, we have $u = v = w = 0$ or $\frac{1}{2}$. For $\Gamma u = 0$ implies $\frac{1}{2}$ and $\frac{1}{4}$ while for $0^1 u$ cannot have the values $0$ or $\frac{1}{2}$ so that in each case only two types are derivable. For $\Gamma u = 0$ implies $u = \frac{1}{2}$ so that only one type corresponds to each of $O^5$ and $O^5$. In all, there are ten types.

$O^1$ derived from $O^1$ parameters

<table>
<thead>
<tr>
<th></th>
<th>$O^1$</th>
<th>$O^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O^0$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$O^1$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$O^1$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$O^1$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$O^1$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$O^1$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$O^1$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$O^1$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>$O^1$</td>
<td>0 0</td>
<td>0 0</td>
</tr>
</tbody>
</table>
1. Equivalent points

We have already deduced all the symmetry relations of the 230 types of point systems. By means of these we can in any system write down the vectors of all the points equivalent to a given point.

If the reduced relations of any system be given by

\[ Y = (-1)^{k_r} h_r A \cdot x A^+ + B_r \]

\[ Y = (-1)^{k_r} h_r A^\pm x A^+ + B_r \]

\[ \ldots \ldots \ldots \ldots \ldots \]

\[ Y = (-1)^{k_r} h_r A^\pm x A^+ + B_r \]

all the points equivalent to \( X \) are given by the relations

\[ Y = (-1)^{k_r} h_r A^\pm x A^+ + B_r \]

where \( h_r \) has all the values \( 1, 2, \ldots, k_r \) for each of the \( n \) values of \( k_r \).

11. The number of such equivalent points in a component set can only depend on the rotational part of the relation, so that in what follows we will deal only with this part.

Now since \( Y = (-1)^{k_r} h_r A^\pm x A^+ \) reduces to \( Y = X \) (except in the cases where \( V_1 = 1 \) and \( k_r = 1 \) or \( 3 \) when it reduces to \( Y = X \)) and there are \( n \) relations of this type, there are in general \( \sum_{k_r}^n \) points equivalent to any point and in all \( \sum_{k_r}^{n+1} \) points in the set.
If there is a rotation of the type \( Yz-X \) there are
\[ \sum_{k}^- n + \frac{1}{2} \]
and if there are of the type \( Yz - A X A \) there are
\[ \sum_{k}^- n + 5m + 1 \] points in the set

Alternatively we may start with the independent reduced relations
\[ Yz = (-1)^n A X A \]
all points equivalent to \( X \) are given by:
\[ Yz = (-1)^n A X A \ldots A X A \ldots A X A \]
where \( h \) has all the values \( 0, 1, 2, \ldots (k-1) \)
The number of equivalent points is in general
\[ k, k, \ldots, k \]
As before the existence of a relation \( Yz-X \) doubles the number of equivalent points; but we need not consider the effect of \( \frac{1}{2} \) \( Yz-A X A \) since it can always be derived from \( Yz-X \) and \( Yz A X A \).
Sometimes the first, sometimes the second method is the most convenient for deriving the equivalent points.

In the following table there is given for each class of rotation the most natural way of doing this.

<table>
<thead>
<tr>
<th>Note</th>
<th>Abbreviations used</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>denotes the relation ( Yz A X A )</td>
</tr>
<tr>
<td>( -A )</td>
<td>&quot; &quot; &quot; &quot; Yz - A X A</td>
</tr>
<tr>
<td>( 1 )</td>
<td>&quot; &quot; &quot; Yz X</td>
</tr>
<tr>
<td>( A )</td>
<td>&quot; &quot; &quot; Yz - X</td>
</tr>
</tbody>
</table>

\( (A_1, A_2, A_3, \ldots, A_k \) (A_k, A_{k-1} \ldots, A_1) \) signifies that every pair-relation in the first bracket is to be combined in turn with each one in the second, and similarly for products with more than two brackets.
<table>
<thead>
<tr>
<th>Symbol of class</th>
<th>Group of rotations</th>
<th>Number of equivalent points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$C_i$</td>
<td>$\pm 1$</td>
<td>2</td>
</tr>
<tr>
<td>$C_3$</td>
<td>1, $-A$</td>
<td>2</td>
</tr>
<tr>
<td>$C_2$</td>
<td>1, $A$</td>
<td>2</td>
</tr>
<tr>
<td>$C_{2h}$</td>
<td>$\pm (1,A)$</td>
<td>4</td>
</tr>
<tr>
<td>$C_{2v}$</td>
<td>1,$A$,$B$,$C$</td>
<td>4</td>
</tr>
<tr>
<td>$Q$</td>
<td>1,$A$,$B$,$C$</td>
<td>4</td>
</tr>
<tr>
<td>$Q_h$</td>
<td>$\pm (1,A,B,C)$</td>
<td>8</td>
</tr>
<tr>
<td>$C_4'$</td>
<td>$1_r$,$r_x$,$r_y$</td>
<td>4</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$1_r$,$A$,$A^{-\frac{x}{2}}$</td>
<td>4</td>
</tr>
<tr>
<td>$C_{4h}$</td>
<td>$\pm (1,A,\pm A,\pm A^{-\frac{x}{2}})$</td>
<td>8</td>
</tr>
<tr>
<td>$D_{2d}$</td>
<td>$(1,\pm r_x,\pm A,\pm A^{-\frac{x}{2}})(1,B)$</td>
<td>8</td>
</tr>
<tr>
<td>$C_{2v}$</td>
<td>$(1,\pm 1,\pm A,\pm A^{-\frac{x}{2}})(1,B)$</td>
<td>8</td>
</tr>
<tr>
<td>$D_{2d}$</td>
<td>$(1,\pm 1,\pm A,\pm A^{-\frac{x}{2}})(1,B)$</td>
<td>8</td>
</tr>
<tr>
<td>$D_{4h}$</td>
<td>$\pm (1,A,\pm A,\pm A^{-\frac{x}{2}})(1,B)$</td>
<td>16</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$1, A, A^{\frac{2}{3}}$</td>
<td>3</td>
</tr>
<tr>
<td>$C_{3h}$</td>
<td>$\pm (1,A, A^{\frac{2}{3}})$</td>
<td>6</td>
</tr>
<tr>
<td>$C_{3v}$</td>
<td>$(1, A, A^{\frac{2}{3}})(1,B)$</td>
<td>6</td>
</tr>
<tr>
<td>$D_2$</td>
<td>$(1, A, A^{\frac{2}{3}})(1,B)$</td>
<td>6</td>
</tr>
<tr>
<td>$D_{3d}$</td>
<td>$\pm (1, A, A^{\frac{2}{3}})(1,B)$</td>
<td>12</td>
</tr>
<tr>
<td>Symbol of class</td>
<td>Group of rotations</td>
<td>Number of equivalent points</td>
</tr>
<tr>
<td>----------------</td>
<td>-------------------</td>
<td>----------------------------</td>
</tr>
<tr>
<td>$C_i$</td>
<td>$(1, A)(1, A, A^\frac{3}{2}, A^\frac{3}{2})$</td>
<td>6</td>
</tr>
<tr>
<td>$D_3$</td>
<td>$\pm (1, A)(1, A, A^\frac{3}{2})$</td>
<td>12</td>
</tr>
<tr>
<td>$C_6$</td>
<td>$1, A^\frac{3}{2}, A^\frac{3}{2}, A, A^\frac{3}{2}, A^\frac{3}{2}$</td>
<td>6</td>
</tr>
<tr>
<td>$C_{6v}$</td>
<td>$(1, A, A^\frac{3}{2}, A, A^\frac{3}{2}, A^\frac{3}{2})(1, B)$</td>
<td>12</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$(1, A, A^\frac{3}{2}, A, A^\frac{3}{2}, A^\frac{3}{2})(1, B)$</td>
<td>12</td>
</tr>
<tr>
<td>$D_{2h}$</td>
<td>$\pm (1, A, A^\frac{3}{2}, A, A^\frac{3}{2}, A^\frac{3}{2})(1, B)$</td>
<td>24</td>
</tr>
<tr>
<td>$T$</td>
<td>$(1, A, B, C)(1, T, T^\frac{3}{4})$</td>
<td>12</td>
</tr>
<tr>
<td>$T_d$</td>
<td>$\pm (1, A, B, C)(1, T, T^\frac{3}{4})$</td>
<td>24</td>
</tr>
<tr>
<td>$O$</td>
<td>$(1, A, B, C)(1, T, T^\frac{3}{4})(1, R)$</td>
<td>24</td>
</tr>
<tr>
<td>$O_h$</td>
<td>$\pm (1, A, B, C)(1, T, T^\frac{3}{4})(1, R)$</td>
<td>48</td>
</tr>
</tbody>
</table>
Coordinates

In general we refer the points of a system to three reference vectors, and the position of any point is given by its coordinates \( \lambda, \mu, \nu \) with respect to these vectors. (In the rhombohedral and hexagonal systems four reference vectors are employed.)

If we write \( X = \lambda a + \mu b + \nu c \); \( Y = \lambda' a + \mu' b + \nu' c \) in the relation \( Y = (-1) \frac{a}{\lambda} X \frac{a}{\lambda} \), it becomes

\[
\lambda' a + \mu' b + \nu' c = (-1) \frac{a}{\lambda} \left( \lambda a + \mu b + \nu c \right) \frac{a}{\lambda} = (-1) \frac{a}{\lambda} a \frac{a}{\lambda} (-1) \frac{a}{\lambda} a \frac{a}{\lambda} \frac{a}{\lambda}
\]

Now \( (-1) \frac{a}{\lambda} a \frac{a}{\lambda} = l_a + m_b + n_c \)
\( (-1) \frac{a}{\lambda} a \frac{a}{\lambda} = l_a + m_b + n_c \)
\( (-1) \frac{a}{\lambda} a \frac{a}{\lambda} = l_a + m_b + n_c \)
where \( l_a, l_b, l_c \) etc. are integers.

Equating coefficients of \( a, b, c \) we have

\[
\lambda' = \lambda^l + \mu^m + \nu^p
\]
\[
\mu' = \lambda^m + \mu^m + \nu^p
\]
\[
\nu' = \lambda^n + \mu^n + \nu^n
\]

The relations between the coordinates of two points connected by a rotation can be expressed in the form of a matrix

\[
\begin{pmatrix}
1, 1, 1, 1
m, m, m, m
n, n, n, n
\end{pmatrix}
\]

Since the inverse of a rotation is a rotation, this matrix must have the property

\[
\begin{pmatrix}
1, 1, 1, 1
m, m, m, m
n, n, n, n
\end{pmatrix} = I
\]

In the following table are given the matrices corresponding to a number of rotations, together with the values of \( \lambda' \mu' \nu' \) in terms of \( \lambda \mu \nu \).
<table>
<thead>
<tr>
<th>Rotation</th>
<th>Matrix</th>
<th>Coordinates</th>
</tr>
</thead>
</table>
| 1        | 1 0 0  
0 0 1  
0 0 1  | λ, μ, ν   |
| -l       | -1 0 0  
0 -1 0  
0 0 -1 | -λ, -μ, -ν|
| a        | 1 0 0  
0 -1 0  
0 0 -1 | λ, -μ, -ν |
| -a       | -1 0 0  
0 1 0  
0 0 1  | -λ, μ, ν  |
| b        | -1 0 0  
0 1 0  
0 0 -1 | -λ, +μ, -ν|
| -b       | 1 0 0  
0 -1 0  
0 0 1  | λ, -μ, ν  |
| c        | -1 0 0  
0 -1 0  
0 0 1  | -λ, -μ, ν |
| -c       | 1 0 0  
0 1 0  
0 0 -1 | λ, μ, -ν  |
| R = [b+c] | 1 0 0  
0 0 1  
0 0 -1 | -λ, ν, μ  |
| -R = [b+c] | 1 0 0  
0 0 -1  
0 -1 0 | λ, -ν, -μ |
| a⁺       | 1 0 0  
0 0 -1  
0 1 0  | λ, -ν, μ  |
| -a⁺      | -1 0 0  
0 0 1  
0 -1 0 | -λ, ν, -μ |
<table>
<thead>
<tr>
<th>Rotation</th>
<th>Matrix</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^+\ T_3$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{bmatrix}$</td>
<td>$\lambda, \varphi, \upmu, \nu$</td>
</tr>
<tr>
<td>$T_3^{-\frac{a+b+c}{2}}$</td>
<td>$\begin{bmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{bmatrix}$</td>
<td>$\nu, \lambda, \mu$</td>
</tr>
<tr>
<td>$a^+\ T_3$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -1 \ 0 &amp; -1 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>$\lambda, -\delta, -\rho, -\mu$</td>
</tr>
</tbody>
</table>
To obtain the coordinates of points equivalent to $\lambda\mu\nu$
in the most general point system we add to the coordinates
of the equivalent points derived by rotations the corresponding
parameters of the general relation

Here

$\lambda' = \lambda + \mu + \nu + u$

$\mu' = m\lambda + m\mu + m\nu + v$

$\nu' = n\lambda + n\mu + n\nu + w$

and similarly for every relation of the system.

The parameters can be taken from the various tables of Ch. VI
the rotations are given in table 7.13 and their matrices in
table 7.22.

For example in the system $q^s$ we have for equivalent points

$\lambda, \mu, \nu; \lambda, -\mu + \frac{1}{2}, -\nu; -\lambda, \mu + \frac{1}{2}, -\nu; -\lambda, -\mu, \nu + \frac{1}{2}$

These will appear in a more symmetrical form if we put

$\mu + \frac{1}{2}$ for $\mu, \nu + \frac{1}{2}$ for $\nu$.

$\lambda, \mu + \frac{1}{2}, \nu + \frac{1}{2}$

$\lambda, -\mu + \frac{1}{2}, -\nu - \frac{1}{2}$

$-\lambda, \mu - \frac{1}{2}, -\nu + \frac{1}{2}$ Here $-\frac{1}{2}$ has been written for $\mu + \frac{1}{2}$

$-\lambda, -\mu - \frac{1}{2}, \nu + \frac{1}{2}$
**Unit cells**

As has been already shown (1.8) we can always choose certain limits for the coordinates of the points in a component set. Such limits mark off a portion of space inside of which all the points of a component set may lie, such a portion is called a unit cell. There is one unit cell corresponding to every point in the skeletal lattice.

Two alternative limits were suggested \( (\lambda, \mu, \nu \leq (B) \frac{1}{2} \leq \lambda, \mu, \nu \leq \frac{1}{2} ) \) \( (B) \) will be used in what follows. The unit cell has thus the shape of a parallelepiped whose centre is at the origin; a point of the set may lie anywhere inside this cell, on three of its faces, three of its edges and one of its angular points. This arrangement is unsymmetrical, but if we make the limits \( -\frac{1}{2} \leq \lambda, \mu, \nu \leq \frac{1}{2} \) so that points may lie on all the faces, edges and corners, such points must count as \( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \) and \( \frac{1}{2} \) of a point respectively which is liable to cause confusion.

The above limits for the unit cell are only useful for the lattices \( \Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}, \overline{\overline{\overline{\Gamma}}} \). If we write the formula of any other \( \gamma \) lattice in terms of its primitive vectors and introduce these limits we obtain unit cells which do not possess the symmetry of the system.

The limits for cells satisfying this condition for each of the 14 lattices, are given with their descriptions, in the following table.
<table>
<thead>
<tr>
<th>Lattice</th>
<th>Limits of cell</th>
<th>Description of cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma' )</td>
<td>(-\frac{1}{2} &lt; \lambda + \mu &lt; \frac{1}{2})</td>
<td>Parallelopipedon</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Oblique rectangular prism</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Rectangular rhombic prism</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Tetragonal prism</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Cube</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Clinodomal prism and pinacoid (Oblique prism on rhombic base)</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Primary rhombic prism (Rectangular prism on rhombic base)</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Rhombic, macro- and brachy- nodal prisms; (Elongated dodecahedron)</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Rhombic dodecahedron</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Rhombic by pyramid and pinacoids (Elongated truncated octahedron)</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Tetragonal bipyramid second order prism and pinacoids.</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Octahedron and cube; (truncated octahedron)</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>Hexagonal prism and pinacoids</td>
</tr>
<tr>
<td>( \Gamma'' )</td>
<td>&quot;</td>
<td>RHOMBOHEDRON</td>
</tr>
</tbody>
</table>
Submultiple sets

If $X$ is such that the relation $Y = (1)A^{\frac{1}{4}}XA^{\frac{2}{4}} + B$ reduces to $Y \equiv X$ then the relation $A^{\frac{2}{4}}B$ does not increase the number of points equivalent to $X$, derived from the other independent relations. The total number of points equivalent to $X$ will accordingly be $1/k$, the number for the most general value of $X$. Such a set of equivalent points is called a submultiple or subset. If in addition another independent relation of the set reduces to $Y \equiv X$ we have a subset, $1/k$, the full set, and we may ultimately arrive at a subset containing only one point.

The relation $Y = (1)A^{\frac{1}{4}}XA^{\frac{2}{4}} + hA/2k + B$ reduces to $Y \equiv X$ in the following cases. If $v = 2$ and $h = 0$ $X = pA + C$ where $A^{\frac{2}{4}}C = -A^{\frac{2}{4}}B = 0$ if $B = 0,$ $C = 0.$ If $v = 1$ and $k \neq 2$ $X = hA/2k + C$ " $A^{\frac{2}{4}}C = -A^{\frac{2}{4}}B = 0$ if $v = 1$ and $k = 2$ $X = hA/4 + C$ " SAC = 0

Expressing this in words; a sub set can be formed when, and only when, its points lie on the axes of rotation, or centre of rotary reflections or reflection planes. (see 5.2 et seq)

For example: The 48 pointer full set of $O_A$ reduces to a 24 pointer sub set when $\mu = \nu$ and when $\lambda = 0$; to a 12 when $\mu = \nu$ and $\lambda = 0$ to an 8 pointer when $\lambda = \mu = \nu$; to a 6 pointer when $\mu = \nu = 0$ and to a 1 pointer when $\lambda = \mu = \nu = 0$.

Consider the case of a multiple point system containing axes of screw symmetry or glide planes which have no rotation axes or reflection planes parallel to them. If a point lie on such an axis or plane the number of points will only be reduced by a rotation included in the screw axis if any such exist.
In no case can the number of points in the sub set reduce to one. All non equivalent points in a multiple system cannot lie on the same screw axes or glide plane. In this case an equation of the type \( Y = A \frac{X}{k} + hA/k \) reduces to \( Y = X + hA/k \) or in the case of a glide plane \( Y = -A \frac{X}{k} + B/2k \) to \( Y = X + B/2 \) which would imply that \( A/k \) and \( B/2 \) were vectors of the skeletal lattice.

In any class of symmetry a subset of one can be formed. All the points of a multiple system however can only form subsets of one in holohedral classes. Similar conditions hold for all other sub sets. If we consider simple systems only we have for each class a sub set that exhibits the symmetry of the class with the least number of points.

In the following table is given a list of sub sets for every class with sub sets characteristic of it. In these classes the last sub set, and in all other classes the full set, is the least sub set. The values of \( \lambda, \mu, \nu \) for each sub set are also given.
<table>
<thead>
<tr>
<th>Class</th>
<th>Sub set</th>
<th>Number in sub set</th>
<th>Class</th>
<th>Sub set</th>
<th>Number in sub set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_i$</td>
<td>$\lambda=\mu=\nu=0$</td>
<td>1</td>
<td>$D_4h$</td>
<td>$\nu=\rho$</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>$\nu=0$</td>
<td>2</td>
<td>$D_4h$</td>
<td>$\nu=\rho$</td>
<td>6</td>
</tr>
<tr>
<td>$C_{4h}$</td>
<td>$\lambda=0$</td>
<td>3</td>
<td>$C_{4h}$</td>
<td>$\lambda=0$</td>
<td>6</td>
</tr>
<tr>
<td>$C_{4h}$</td>
<td>$\mu=\nu=0$</td>
<td>4</td>
<td>$D_{4h}$</td>
<td>$\nu=0$</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>$\lambda=\mu=\nu=0$</td>
<td>1</td>
<td>$D_{4h}$</td>
<td>$\lambda=0$</td>
<td>12</td>
</tr>
<tr>
<td>$D_{4d}$</td>
<td>$\lambda=0$</td>
<td>4</td>
<td></td>
<td>$\nu=0$</td>
<td>12</td>
</tr>
<tr>
<td>$C_{4h}$</td>
<td>$\mu=\nu=0$</td>
<td>4</td>
<td></td>
<td>$\lambda=\nu=0$</td>
<td>6</td>
</tr>
<tr>
<td>$D_{4h}$</td>
<td>$\lambda=0$</td>
<td>8</td>
<td></td>
<td>$\mu=\nu=0$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$\mu=0$ or $\mu=\nu$</td>
<td>8</td>
<td></td>
<td>$\lambda=\mu=\nu=0$</td>
<td>1</td>
</tr>
<tr>
<td>$T_h$</td>
<td>$\lambda=0$</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_d$</td>
<td>$\mu=\nu$</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$O_h$</td>
<td>$\mu=\nu$</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda=0$</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mu=\nu$, $\lambda=0$</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda=\mu=\nu$</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mu=\nu=0$</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda=\mu=\nu=0$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Available positions

The points of a sub set must lie either on rotation axes or reflection planes. There are thus in every point system a finite number of planes, lines and points in which the points of sub sets must lie. All the points of the component set may be located there only in those classes in the foregoing table. The case of one point sub sets is peculiar; in that only a limited number of such sub sets can co-exist in any system. Only a certain number of systems can reduce to one point sub sets, other systems can only reduce to two pointers and so forth.

(At this point it was hoped to be able to introduce a table showing every sub set possible in each of the 230 systems for the together with the available positions of such points of such sub sets.)

This table is given in the second form of the paper. 223.
CHAPTER VIII

PLANES

1. The equation to a plane

If the variable vector \( \mathbf{X} \) be expressed in terms of the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) then the so that \( \mathbf{X} = p \mathbf{a} + q \mathbf{b} + r \mathbf{c} \); then the existence of an equation of the type

\[
fp + gq + hr + j = 0
\]

between the variable scalars \( p, q, r \) and the constant scalars \( f, g, h, j \) indicates that the extremity of \( \mathbf{X} \) must always lie on a plane determined by \( f, g, h, j \).

If \( \mathbf{X} \) is a vector of a point system the values of \( p, q, r \) are restricted. All permissible values of \( p, q, r \) which satisfy the equation \( fp + gq + hr + j = 0 \) indicate points of the system lying in the plane \( f, g, h, j \). Should any such points exist, the number of them is given by the number of permissible solutions of the above equation.

2. The law of rational indices

\( p, q, r \) can always be put in the form \( l + \lambda \), \( m + \mu \), \( n + \nu \); where \( l, m, n \) can take all integral values \( \frac{1}{z} < \lambda < \frac{1}{z} \) and \( z \) is finite.

If any point corresponding to \( \lambda, \mu, \nu \) be on the plane we have

\[
f(l + \lambda) g(m + \mu) h(n + \nu) + j = 0
\]

or

\[
fl + gm + ln + j = 0
\]

where \( j = f \lambda + g \mu + h \nu + j \)

this is an equation which only admits of integral solutions, there will be a doubly infinite set of these if, and only if \( f, g, h, j \) are all rational multiples of the same irrational number.
By multiplying by a suitable factor \( f, g, h \) may be reduced to integers with no common factor; in this form \( f, g, h \) are called the indices of the plane. In the subsequent work we are only concerned with such planes; the term plane will refer exclusively to planes containing a doubly infinite system of points, corresponding to a given point of the three dimensional point system. As all points form lattices, we will in the following sections consider lattices only.

3 Formula of a plane net

The points of the lattice /la+mb+nc which lie in the plane \( fl+gm+hn+j=0 \) form a two dimensional net. We can obtain the formula of this net by eliminating \( l, m, \) or \( n \) from the formula of the lattice by means of the equation of the plane. The following procedure is however more symmetrical.

Let \( u_1, v_1, w_1; u_2, v_2, w_2 \) be two points on the plane \( fl+gm+hn=0 \) and \( u_1, v_1, w_1 \) be a point on the plane \( fl+gm+hn+j=0 \) the general solution to the equation of the plane may now be written

\[
\begin{align*}
l &= s u_1 + tu_2 + u_3 \\
m &= s w_1 + tw_2 + v_3 \\
n &= s w_1 + tw_2 + w_3
\end{align*}
\]

Introducing these values into the formula of the lattice we obtain the formula of the net as

\[
/ (s u_1 + tu_2 + u_3) a + (s v_1 + tv_2 + v_3) b + (s w_1 + tw_2 + w_3) c
\]

or

\[
/s(u_1 a + v_1 b + w_1 c) + t(u_2 a + v_2 b + w_2 c) + (u_3 a + v_3 b + w_3 c)
\]

writing \((u_1 a + v_1 b + w_1 c) = e_1; u_2 a + v_2 b + w_2 c = e_2; u_3 a + v_3 b + w_3 c = e_3\) the formula becomes

\[
/se_1 + te_2 + e_3
\]
If the plane passes through the origin $e, e^2 = 0$ and the formula of the net reduces to \( /se, + te, \)

$e, e^2$ will be primitive vectors of the net if $e, e^2, e, e^2$ are for the system lattice. That is if

\[
\begin{vmatrix}
  u, & v, & w, \\
  u^2, & v^2, & w^2 \\
  u, & v, & w, \\
\end{vmatrix} = \pm 1
\]

If we substitute for the first column, that formed by multiplying the columns by $f, g, h$ respectively and adding, we have

\[
\begin{vmatrix}
  0, & v, & w, \\
  0, & v^2, & w^2, \\
  -1, & v, & w, \\
\end{vmatrix} = \pm f, g, h
\]

Expanding

\[v, w^2 - v^2, w = \pm f, g, h\]

Similarly

\[w, u - w, u = \pm g, h\]

\[u, v - u, v = \pm h, j\]

Since $\pm u, v, w$ are all integers and $f, g, h$ have no common factor $j = 1$, but the condition that $e, e^2$ and $e, e^2$ should be primitive is independent of the value of $e, e^2$ and consequently of $j$; this condition as therefore

\[v, w^2 - v^2, w = \pm f, g, h\]

\[w, u - w, u = \pm g, h\]

\[u, v^2 - u, v^2 = \pm h, j\]

\section{Parallel planes}

Planes with the same values of $f, g, h$ but with different values of $j$ are parallel. If the indices are expressed in terms of primitive vectors $j$ must be an integer and as no plane can lie between the parallel planes $f, g, h, j$ and $f, g, h, (j + 1)$, these are called successive planes.
5 Distance between planes

The perpendicular distance from the origin to the plane

\[ f + gm + hn - l = 0 \]

may be expressed as the quotient

\[ \frac{\text{Volume of } \parallel \text{pipedon } e, e, e, e}{\text{Area of } \parallel \text{gm } e, e, e} \]

\[ = \frac{S_{e,e,e}}{TV_{e,e}} \]

\[ = \frac{S(u_a + v_a + b + w, c)(u_a + v_a + b + w, c)}{TV(u_a + v_a + b + w, c)(u_a + v_a + b + w, c)} \]

\[ \frac{u, v, w, S_{abc}}{TV^2(v, w - v, w)V_{bc}} \]

Now \( u_a \) can always be chosen so that \( (u_a, v_a, w_a) = 1 \)

also \( (v, w - v, w) = 1 \) etc. We have therefore

Perpendicular distance \( D = \frac{S_{abc}}{T(fVbc + gVca + hVab)} \)

This expression may of course be derived directly from the
equation of the plane and is true for all real values of \( f, g, h, j \)

In its most general form it is

\[ D = \frac{\sqrt{S_{abc}}}{T(fVbc + gVca + hVab)} \]

This expression though simple is not convenient for
numerical calculation as it involves sums and products of vectors
It can easily be transformed however into one that is.
We make use of the formula \( -(Sabc) = S \cdot VbcVcaVab \) (see Joly "Manual of Quaternions" Art. 25, Ex. 4)

\[-(Sabc) = S \cdot VbcVcaVab\]

\[-= S(bc-bc)(ca-ca)(ab-Sab)\]

\[-= S bcccaab-2Sbc.caab+2SbcSca.ab-3SbcScaSab\]

\[-=abc-aab(Sbc)+3SbcScaSab+3SbcScaSab\]

\[-=abc+a(Sbc)+b(Soa)+c(Sab)-2SbcScaSab\]

Also \( f(fVbc+gVca+hVab) = f(Vbc)^2 + 2ghS \cdot VcaVab \)

\[-=2ghVbc-Sbc-ScaSab\]

We have therefore

\[D = \frac{abc+2Sbc+3SbcScaSab}{f(Vbc)^2 + 2ghSbc-ScaSab}\]

Now if we put \( a \) for the scalar \( Ta, \cos A \) for \( S .UbUc \); \( \sin A \) for \( TV.UbUc \) and similarly for \( b \) and \( c \); \( B \) and \( C \) the expression takes a form involving scalars only

\[D = \frac{abc(1-\cos A-\cos B+2\cos A \cos B \cos C)}{2gh \sin A - 2gh (sca - cos A \cos B \cos C)}\]

Finally we obtain the familiar formula

\[D = \sqrt{\frac{1-\cos A + 2\cos A \cos B \cos C}{a+b+c}}\]

This formula loses much of its complexity when applied to symmetrical lattices. It is given for each of the 14 lattices in the table on p 94.
So far we have only considered lattices whose formula is 
written \( /la+mb+no \). It is not always convenient to use such a 
formula. For all lattices except \( \mathcal{J}_h, \mathcal{J}_{1h}, \) the formula may 
be written 
\[
\frac{(d_l^2 + q_m + q_n) + 1}{2} (d_l^2 + q_m + q_n) + b + \frac{(d_l^2 + q_m + q_n) + c}{2}
\]
(for \( d_l, d_m, d_n, d_l^2 = e, q_n = e, q_n = e \) and \( d_l^2 \) \( d_m^2 = 0 \))
The equation of the plane becomes 
\[
f \frac{(d_l^2 + q_m + q_n) + 1}{2} (d_l^2 + q_m + q_n) + b + \frac{(d_l^2 + q_m + q_n) + c}{2} = 0
\]
f, g, h may be such that each of the first three terms is an integer 
for all values of \( l, m, n \).
In this case \( j \) is an integer and successive planes are at the 
same distance as in the corresponding lattice \( /la+mb+no \)
If the first three terms do not reduce to integers \( j \) must be 
of the form \( k/2 \) when \( k \) is an integer and successive planes are 
half as far apart as in the corresponding lattice.
If \( d_{jk} \) stands for the distance between successive planes parallel 
to \( fp + qa + hr = 0 \) in any lattice, and \( D_{jk} \) for the distance in a 
corresponding lattice of formula \( /la+mb+no \), then 
\[
d_{jk} = \frac{D_{jk}}{\delta_{jk}}
\]
where \( \delta_{jk} \) depends both on the lattice and on \( f, g, h \) and is always 
1, \( \frac{1}{3} \) or in the case of \( \mathcal{J}_h, 1/3 \). The value of \( \delta_{jk} \) for all lattices 
and values of \( f, g, h \) is given in the table on p 94.
The case of \( \rho_\alpha \) and \( \rho_\delta \) must be dealt with separately; in this case four reference vectors are used and the equation of the plane becomes \( fp + gq + hr + j = 0 \) where \( ghj \) are connected by the relation \( g + h + i = 0 \).

\[ fp + gq + hr + j = 0 \] is the equation of the same plane referred to the three vectors \( a, b, c \) Now applying the formula for \( D \) we have

\[
D = \frac{b^2}{\sqrt{f a^2 (g^2 + h^2 + i^2)}}
\]

since \( \cos A = \frac{1}{2} \) \( \cos B = \cos C = 0 \)

\[
= \frac{1}{\sqrt{f a^2 \frac{1}{3} (g^2 + h^2 + i^2)(g + h + i)}}
\]

since \( g + h + i = 0 \)

\[
= \frac{1}{\sqrt{f a^2 \frac{1}{3} (g^2 + h^2 + i^2)}}
\]

since \( g^2 + h^2 + i^2 \)

\[
= \frac{1}{b}
\]

and

\[
\frac{1}{g + h + i}
\]

The formula of \( \rho_\alpha \) is

\[
\frac{1}{la + (2m - n - o)b + (m + 2n - o)c + (m - n + 2o)d} \]

\[
g(2m - n - o)/3 + h(-m - 2n - o)/3 + i(-m - n + 2o)/3 \] is an integer when \( m(2g - h + i)/3 + n(-g + 2h - i)/3 + c(-g - h + 2i)/3 \) is an integer

that is \( 2g - h - j \) is divisible by 3. In this case = 1 and in all other cases = 1/3 But 2g - h + i = 3g and is therefore always divisible by 3 so that for this lattice \( \delta = 1 \)

The formula of \( \rho_\delta \) is

\[
\frac{1}{a + (m + n)b + (m - n)c + (n - l)d} \]

by similar reasoning we can show that if \( f + h + i \) is a multiple of 3 \( \delta = 1 \) and if not \( \delta = 1/3 \)
Forms

We now return to the most general point system.

The plane parallel to \( f' + gm + hn = 0 \) through the point \( \lambda, \mu, \nu \) is \( fp + gq + hr + j = 0 \), where \( j \) (no longer necessarily an integer) is given by \( f\lambda + g\mu + h\nu + j = 0 \).

If three points \( \lambda_1, \mu_1, \nu_1; \lambda_2, \mu_2, \nu_2; \lambda_3, \mu_3, \nu_3 \) lie in the plane \( f, g, h, j \), the three points connected with them by the rotation \( Y = (1^l A, X A, 1^j) \) \( \lambda_1', \mu_1', \nu_1'; \lambda_2', \mu_2', \nu_2'; \lambda_3', \mu_3', \nu_3' \): WILL lie in the plane \( f', g', h', j' \) where \( f', g', h', j' \) are given by the relations

\[
\begin{align*}
f' &= l_1 f + l_2 g + l_3 h \\
g' &= m_1 f + m_2 g + m_3 h \\
h' &= n_1 f + n_2 g + n_3 h
\end{align*}
\]

where \( l_1, l_2, l_3 \), etc., are the minors of \( l_1, l_2, l_3 \), in the matrix \( (l_1, m_2, n_3) \) of the rotation. In other words \( (l_1, m_2, n_3) \) is the matrix of the inverse rotation \( Y = (1^l A, X A, 1^j) \). For substituting for \( \lambda' \) and \( f' \) in terms of \( \lambda, f \) and \( f' \) we have

\[
\begin{align*}
\xi \lambda f l \lambda + \xi m f m \lambda + \xi n f n \lambda + j &= 0 \\
-j(\xi l_1 l_1 + \xi m_2 m + \xi n_3 n) - \xi g f (l_1 l_1, m_1 m, n_1 n) &- \xi h f (l_1 l_1, m_2 m, n_3 n) + j' &= 0
\end{align*}
\]

This is the case only if \( \xi l_1 l_1 = 1 \) and \( \xi l_1 l_1 = 0 \) and since

\[
(l_1, m_2, n_3) = l_1 \quad l_1 = m_2, n_2, m_2, n_2
\]

The planes \( f, g, h, j \) and \( f', g', h', j' \) are said to be equivalent planes.
If the symmetry relation is of the type \( Y = (l') x + a + b + w \)
the plane equivalent to \( f, g, h, j \) is \( f', g', h', j' \) where \( f, g, h \) are
as \( \Xi \) before but \( j' = fu + gv + hw + j \).

61. All the planes equivalent to a given plane are said to
constitute a form. A form has the same number of planes as the
number in the full set of equivalent points of the system.
For certain values of \( f, g, h \) however \( k \) planes of the form coincide,
and a sub form results each of whose planes contains \( k \) equivalent
points, unless these points themselves form a sub set. (planes
such planes must be perpendicular to rotation axes or reflection.
The same considerations that applied to the limitation of
points in component sets apply to forms also, except that in this case \( j \) alone has to be limited.

62. If the system contains no screw axes or glide planes to which
there are no parallel rotation axes or reflection planes, the
form is called an infinite form; otherwise it is a finite form.
Infinite forms may in certain cases contain up to 6 parallel
planes. Consider for example the plane \( 1, 0, 0, 0 \) perpendicular
to the screw axis \( Y = \frac{a}{k} + a/k \). The planes contained in the
form are \( 1, 0, 0, 0 ; 1, 0, 1/k ; 1, 0, 0, 2/k ; \ldots 1, 0, 0, (k-1)/k \)
al of which are parallel.

By the symmetry of the system all the planes of a form must
be congruent.
Densities of systems and planes

The point density of a system is defined as the number of points per unit volume of the system; alternatively as

\[ \frac{\text{Number of points in unit cell}}{\text{Volume of unit cell}} \]

The number of points in a unit cell is simply the number \( N \) of points in a component set; the volume of a unit cell is \( S_{abc} \) when \( a, b, c \) are primitive vectors, in general however it is

\[ \Delta S_{abc} \]

where \( \Delta = \begin{vmatrix} 1,1,1 \end{vmatrix} \) and \( \begin{vmatrix} 1 \end{vmatrix} \) are the coordinates of \( a, b, c \)

in terms of primitive vectors.

The values of \( \Delta \) for each of the 14 lattices is given in the table on p 94. We have therefore as an expression for the point density

\[ \frac{\Delta N}{S_{abc}} \]

The point density of a plane may be defined as

\[ \frac{\text{Number of points in the plane per unit cell}}{\text{Area of unit cell}} \]

If \( n \) is the number of points of a component set that lie in the plane and \( \delta \) and \( \Delta \) have the same meaning as in the preceding sections; Point density

\[ \frac{n}{S_{abc}} \]

\[ = \frac{\delta \Delta n}{\overline{T(FVbc+eVoa+HVab)}} \]

Weighted densities

In a multiple system we may attribute to each point a weight factor \( m_r \); \( m_r \) being the same for all equivalent and corresponding points
The weighted density of a system or plane is the weight per unit area-volume and per unit area respectively. If \( N \) and \( n \) are the number of points of the same kind in a component set of the system and the plane respectively, the weighted densities are

\[
\Delta \xi_{\text{N or } \text{n}} \quad \text{Sabc} \\
\text{and} \quad \frac{\xi \Delta \xi_{\text{N or } \text{n}}}{\mathcal{T}(fVbc+gVca+hVab)}
\]

In the following table are given a number of important constants and expressions for each of the 14 lattices. From these by the use of the formulae reproduced below we can calculate the perpendicular distance between successive parallel planes and the densities of systems and planes.

Distance from origin to a plane \( fp+gq+hr+j=0 \)

\[
\frac{\xi \text{Sabc}}{\mathcal{T}(fVbc+gVca+hVab)}
\]

Distance between successive planes parallel to \( fl+gm+hn=0 \)

\[
\frac{\xi \text{Sabc}}{\mathcal{T}(fVbc+gVca+hVab)}
\]

Volume of unit cell = \( \frac{1}{\Delta} \text{Sabc} \)

Density of points in a system = \( \frac{\Delta N}{\text{Sabc}} \)

Density of points in a plane

\[
\frac{\xi \Delta n}{\mathcal{T}(fVbc+gVca+hVab)}
\]
Lattice

\[ T(\tau \rho \nu + \sigma \tau \pi + \eta \nu \pi) \]

\[ \Delta \gamma \]

1 1 1 \( \text{abc} \sqrt{1 - \cos^2 A \cos^2 B \cos^2 C} \)

1 1 1 \( \text{abc} \cos A \cos B \cos C \)

1 1 \( \text{abc} \sin A \)

2 \( \text{f+g even} \)

\( \frac{1}{2} \text{f+g odd} \)

2 \( \text{g+h even} \)

\( \frac{1}{2} \text{g+h odd} \)

4 \( \text{f+g+h all odd} \)

\( \frac{1}{2} \text{otherwise} \)

2 \( \text{two of f, g, h even, one odd} \)

\( \frac{1}{2} \text{otherwise} \)

1 \( \text{ab} \)

\( \frac{1}{2} \text{otherwise} \)

2 \( \text{two of f, g, h even, one odd} \)

\( \frac{1}{2} \text{otherwise} \)

1 \( \text{a}^3 \)

\( \frac{1}{2} \text{otherwise} \)

2 \( \text{two of f, g, h even, one odd} \)

\( \frac{1}{2} \text{otherwise} \)
We may consider $\Gamma_{\alpha}$ to be expressed in terms of the three primitive vectors $1/3, 0, 1/3; -1/3, 1/3, 0; 1/3, -1/3, 0$ in which case its formula becomes $/1a+mb+nc; \Delta=1, S=1$

$$S_{abc} = a_1 \sqrt{1 - 3 \cos^2 A + 2 \cos^3 A}$$

since here $a = b = c$

$$A = B = C$$

$$T(fVbc+gVoa+hVab) = a \sqrt{(f^2g^2h^2)\sin^2 A, \Delta + 2(gh+hf+fg)(\cos^4 A - \cos A)}$$
CHAPTER IX

X-Ray analysis of Crystals

(Elementary)

We may consider a crystal as in general a multiple point system in which the place of the points is taken by atoms of the different elements.

A crystal is always finite and is in general bounded by planes, but the number of atoms in even the smallest crystal is so great that these limitations need not be taken account of in considerations of internal structure.

The external symmetry of the forms of a crystal are often sufficient to determine which of the thirty-two classes of symmetry it belongs to, but cannot possibly reveal, except in Class 0, which skeletal lattice and point system is the basis of their structure.

1 Defraction of X-rays by crystals.

The distance between the atoms of a crystal is comparable to the wave length of X-rays which are diffracted by a crystal as from a three-dimensional grating.

Bragg has considered the effect as that of reflection from successive, equally-spaced, congruent planes of the crystal and obtains the simple grating formula
\[ n\lambda = 2d \sin \theta \]

for maximum reflection where

- \( d \) is the distance between successive planes
- \( \theta \) is the angle which the incident and reflected rays make with the plane
- \( \lambda \) is the wave-length of the X-ray, and
- \( n \) is an integer fixing the order of the reflection.

**11** If we keep \( \lambda \) constant and vary \( \theta \) we obtain maxima of reflection when \( \theta = \theta_i, \theta_2, \ldots, \theta_r \) given by:

\[
\sin \theta_i = \frac{n\lambda}{2d}
\]

\[
\sin \theta_2 = \frac{2n\lambda}{2d}
\]

\[
\vdots
\]

\[
\sin \theta_r = \frac{r n \lambda}{2d}
\]

**12** On the other hand, if we keep \( \theta \) constant and vary \( \lambda \), or what is the same thing, use "white" light, we have reflected at an angle \( \theta \) rays of wave-length \( \lambda, \lambda_2, \ldots, \lambda_r \) given by:

\[
\lambda_i = 2d \sin \theta
\]

\[
\lambda_2 = \frac{2d}{2} \sin \theta
\]

\[
\vdots
\]

\[
\lambda_r = \frac{2d}{r} \sin \theta
\]

These two cases are the basis of the Bragg and Laue methods respectively.
Distribution of Energy among orders of Reflection.

In general, however, the series of planes parallel to a given plane are not all congruent but consist of sets of different planes which are repeated regularly. Bragg has shown that the effect of this is analogous to that of the form of grating lines and leads to a different distribution of intensity among the orders of reflections than that which would result from a series of congruent, equally-spaced planes.

In the latter case the intensity of the first, second, etc. orders falls off approximately as the inverse square of their order, i.e. in the ratio: 1, \( \frac{1}{4} \), \( \frac{1}{9} \), etc.

The intensity of a reflection of order \( n \), due to a distribution of planes at distances \( d_1, d_2, \ldots, d_n \) from an initial plane, the whole being repeated at a distance \( d \) is given by

\[
I_n = \frac{k}{n^2} \left( m_1 + m_2 \cos \frac{2n \pi d}{d} + m_3 \cos \frac{2n \pi d}{d} + \ldots \right)^2
\]

\[
(\sin \frac{2n \pi d}{d} + m_2 \sin \frac{2n \pi d}{d} + \ldots)^2
\]

where \( m_1, m_2, m_3 \) represent the reflecting powers of the successive planes. \( m_1 \) is roughly proportional to the electronic density of the plane, i.e. to the weighted density, if to each atom of the plane we attribute the number of electrons associated with it, i.e. its atomic number. As the factor \( k \) is unknown we require relative and not absolute electron densities.
The Bragg X-ray Spectrometer Method.

In the Bragg method the crystal is mounted on a goniometer and a very fine beam of monochromatized X-rays directed on to one face. The intensity of the reflected beam is measured by an ionisation chamber and a sensitive electrooscope. The crystal and the ionisation chamber are so placed that the latter is always at twice the angular distance from the incident beam as the crystal face. The angles of reflection and the intensities of the first few order reflections from the face are measured.

Determination of skeletal lattice

Such measurements for three faces of known indices suffice, for if \( \lambda \) is known, to determine absolutely the skeletal lattice of the system, i.e. to give both the nature of the lattice and the lengths and mutual angles of its reference vectors.

For by the formula given above we have

\[
d = \frac{n \lambda}{\sin \theta}
\]

also

\[
d = \frac{\sum S_{abc}}{T(\sum f_{\beta} + g_{\gamma} + h_{\delta})}
\]

\[
= \frac{\sum 1 - \cos \alpha}{\sum f_{\alpha} \sin \alpha + \sum \cos \beta \cos \gamma}
\]

Now we know \( \lambda \) and \( \sin \theta \); also from crystallographic data we know \( A \), \( B \), \( C \) and \( f \), \( g \), \( h \).
Hence we have three equations of the type

\[ a_1x^2 + b_1y^2 + c_1z^2 + f_1yz + g_1zx + h_1xy + d_1 = 0 \]
\[ a_2x^2 + b_2y^2 + c_2z^2 + f_2yz + g_2zx + h_2xy + d_2 = 0 \]
\[ a_3x^2 + b_3y^2 + c_3z^2 + f_3yz + g_3zx + h_3xy + d_3 = 0 \]

These are always soluble (in the most general case only by elliptic functions). Thus \( a, b, c \) can always be found.

In practice, however, it is found that only the planes with simple indices give good reflections and these are generally used. In this case the equations above reduce to one or two terms each and the solution is correspondingly easy.

For the factor \( \delta \) (see § 32) which depends upon the lattice and upon the values of \( f, g, h \), the latter are so chosen as to afford a criterion of the particular lattice.

This may necessitate measurements from more than three planes.

Examples of planes which can be used for this criterion in the various crystal systems are given in the following table. For each lattice in a given system there will exist a ratio connecting the \( d \)'s of the various planes. A sufficient number of planes are chosen so that no two lattices of the same system have the same ratios of their \( d \)'s.
<table>
<thead>
<tr>
<th>Lattice Type</th>
<th>Formula</th>
<th>Formula</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triclinic</td>
<td>( \frac{a}{\sin A} \sqrt{1 - 2 \cos A + 2 \cos A \cos B \cos C} )</td>
<td>( \frac{b}{\sin B} )</td>
<td>( \frac{c}{\sin C} )</td>
</tr>
<tr>
<td>Monoclinic</td>
<td>( \frac{d_{100}}{a} )</td>
<td>( \frac{d_{110}}{b \sin A} )</td>
<td>( \frac{d_{111}}{c \sin A} )</td>
</tr>
<tr>
<td>Orthorhombic</td>
<td>( \frac{d_{100}}{a} )</td>
<td>( \frac{d_{110}}{b} )</td>
<td>( \frac{d_{111}}{c} )</td>
</tr>
<tr>
<td>Tetragonal</td>
<td>( \frac{d_{100}}{a} )</td>
<td>( \frac{d_{110}}{b} )</td>
<td>( \frac{d_{111}}{b} )</td>
</tr>
<tr>
<td>Cubic</td>
<td>( \frac{d_{100}}{a} )</td>
<td>( \frac{d_{110}}{a \sqrt{2}} )</td>
<td>( \frac{d_{111}}{a \sqrt{3}} )</td>
</tr>
<tr>
<td>Rhombohedral</td>
<td>( \frac{d_{100}}{a} )</td>
<td>( \frac{d_{101}}{\sqrt{3} b / 2} )</td>
<td>( \frac{d_{111}}{2 \sqrt{3} b / 6} )</td>
</tr>
<tr>
<td>Hexagonal</td>
<td>( \frac{d_{100}}{a} )</td>
<td>( \frac{d_{111}}{2 b} )</td>
<td></td>
</tr>
</tbody>
</table>
Thus in the case of rock salt NaCl a cubic crystal
\[ d_{001} : d_{110} : d_{111} = 1 : 1/\sqrt{2} : 2/\sqrt{3} \]
The skeletal lattice is therefore \( \Gamma' \) and \( a = 2d_{100} \)

It should be remembered in this connection, that except in
the cubic system that the indices assigned to faces by
crystallographers are not necessarily the same as those which
are derived from the actual lattice of the crystal. As the
former indices must necessarily be known first a lattice may
be deduced which is not one of the 14 types. In such a case
it is always possible to change the system of indices so as to
reduce the lattice to a recognised form. This is well illustrated
by the case of calcite, a crystal of class \( \Gamma \) for which Bragg found
a face centered rhombohedral lattice with faces parallel to the
form 101-1. If the lattice is taken as \( \Gamma' \), the indices of this
form become 12-1-1.

Screw axes

If the reflections from planes perpendicular to screw axes
are measured the value of \( d \) will be \( h/k \) its normal value where
\( h/k \) is the pitch of the screw. Since screw axes can only exist
perpendicular to certain \( \{hkl\} \) planes it is always possible by
choosing suitable planes to avoid any error on this score, and at
the same time to determine the position and nature of all screw
that exist.

Thus in the case of the diamond a cubic crystal containing
only carbon atoms, \( d_{001} : d_{110} : d_{111} \) is \( 1 : 1/\sqrt{2} : 2/\sqrt{3} \) a ratio not charac-
teristic of any lattice and indicating a screw axis \( \alpha \).
The real ratio should be \( 1 : 1/\sqrt{2} : 2/\sqrt{3} \) characteristic of \( \Gamma' \).
Number of points in a component set

Having determined the lattice absolutely we can proceed to find the number of points in each component set by comparing the density of the crystal on the assumption that there are $N$ molecules per unit cell, with the observed density of the crystal.

The weight of each molecule is $Mm$ where $M$ is its molecular weight and $m$ the weight of a hydrogen atom. We have therefore,

$$\text{Density} = \frac{\Delta NMm}{SabO} \quad \text{or} \quad N = \frac{\rho SabO}{\Delta Mm}$$

The formula of a molecule is in general written $X_1 Y_2 Z_3 \ldots$ where $x, y, z \ldots$ represent the numbers of atoms of $X, Y, Z$ respectively in the molecule. The number of atoms of each kind per unit cell is accordingly $N_x, N_y, N_z \ldots$ The number $N_x$ when $x$ is the least of $x, y, z, \ldots$ is of the greatest importance in the elucidation of crystal structure. It at once limits the number of classes and systems that the crystal may belong to. The crystal can belong to no system whose least subset has more points per unit cell than $N_x$.

Thus in the case of quartz $SiO_2$ a crystal of the class $D_3$, measurements show that the lattice is $\Gamma$ with $a = 5.38 \times 10^{-5}$ cm.

We have therefore for $N$

$$N = \frac{\rho SabO}{2\Delta Mm}$$

$$= 2.96 \approx 3$$

From measurements of $d_{1000}$ we have $d_{100} = a/3$ and therefore there must be a screw axis. The basic system of quartz must therefore be one of $D_3^3, D_3^4, D_3^5, D_3^6$ with the Si atoms as a 3-point subgroup.
Determination of class

The class of a crystal can usually be determined by crystallographic methods. In cases of crystals which only develop very simple forms the information these afford has to be supplemented by optical, electrical or other methods. By the Bragg method it is possible to detect directly all symmetry relations except those that involve a centre of symmetry.

If the class of the crystal is merohedral the orders of reflection from faces which belong to the same holohedral but not to the same merohedral forms will in general give different intensity distributions. The method fails when the two faces considered belong to forms whose faces are parallel to each other. For in this case the spacing of planes differ only in the two faces in that the sequence is inverse and as can be seen from the formula \[ \sum_i \alpha_i \frac{1}{d_i} \] changing the signs of \[ d_1, d_2, d_3, \ldots \] does not affect the reflection intensities.

Thus in the case of zinc blende ZnS which is known to belong to class \( T_d \) there is no difference in the reflections from faces of the different forms \( 1,1,1 \) and \( -1,-1,-1 \), while for pyrites FeS\(_2\) which belongs to class \( T_h \) the reflections from the faces 210 and 120 are totally distinct.

For the final determination of class in these doubtful cases we must have recourse to the methods of the next section.
Determination of atomic positions

We may write the formula of the whole crystal as

\[ \alpha \left\{ \begin{array}{c}
\begin{array}{c}
\lambda_{X}^1 \, f_{X}^1 \\
\lambda_{X}^2 \, f_{X}^2 \\
\vdots
\end{array}
\end{array} \right\} + \beta \left\{ \begin{array}{c}
\begin{array}{c}
\lambda_{Y}^1 \, f_{Y}^1 \\
\lambda_{Y}^2 \, f_{Y}^2 \\
\vdots
\end{array}
\end{array} \right\} + \gamma \left\{ \begin{array}{c}
\begin{array}{c}
\lambda_{Z}^1 \, f_{Z}^1 \\
\lambda_{Z}^2 \, f_{Z}^2 \\
\vdots
\end{array}
\end{array} \right\}
\]

where \( \lambda_{X}^i \) are the coordinates of an atom of \( X \),

\( \lambda_{Y}^i, \lambda_{Z}^i \) are the coordinates of all equivalent atoms of \( X \),

\( \lambda_{k, X}^i \) are the coordinates of all the atoms in a component set.

If there are \( N \) atoms.

There are \( N^2 \) atoms in a component set we have after determining

the lattice, \( 3N^2 \) unknowns to find, before the whole crystal is

absolutely determined. We may have already found, however, a number

of relations connecting these coordinates. If we know the symmetry

class, the number of unknowns is reduced to \( 3N^2 / k \) where \( k \) is the

number of equivalent points in the class. If there is a sub set

some of the unknowns may be exactly determined. We know what screw

axes exist and from this and the information of the sub sets we

may be able to limit the number of possible systems that the crystal

may belong to, a very few. In this case, for each possible system

the number of unknowns reduces to \( 3^2 \) and may even from consider-

ations of sub sets reduce to 0 in which case the crystal is

completely determined. Whether such reductions may be affected

or not the following method is applicable.

Consider the series of planes parallel to \( fl + gm + hn = 0 \). One

of each of passing through the points \( \left\{ \begin{array}{c}
\begin{array}{c}
\lambda_{X}^i \\
\lambda_{Y}^i \\
\lambda_{Z}^i
\end{array}
\end{array} \right\} \). Their point densities

are all equal to the density of points in the corresponding net

of the skeletal lattice, while their electron densities are

simply proportional to the atomic numbers of \( X, Y, Z, \ldots \).
We have therefore for the value of the $m_r$ of the formula 9.21

$$m_r = \frac{\delta \Delta x}{T(fVbc + gVca + hVab)}$$

where $n_x$ is the atomic number of an element X.

The spacing of the planes is given by

$$d_{n_x}/d = j_{n_x}/\delta$$

where $j_{n_x} = f\lambda_{n_x} + g\mu_{n_x} + h\nu_{n_x}$.

Substituting in the formula

$$I_r = \frac{\delta \Delta k}{nt(fVbc + gVca + hVab)} \left[ \left\{ \frac{\gamma n_x \cos 2\pi \nu_{n_x}/\delta}{\delta^2} \right\}^2 + \left\{ \gamma \sin 2\pi \nu_{n_x}/\delta \right\}^2 \right]$$

If we are able to measure reflections from P planes and can obtain p orders of reflection from each the r th plane we have $\Sigma p_r$ equations of this type. However as the equations are transcendental in $j_{n_x}$ we cannot tell how many would be needed to find the $3N_x$ unknowns. The solution of a number of such equations is except in some simple cases extremely difficult and laborious. It should not prove difficult to construct a machine somewhat similar to a harmonic analyser, which would make $\left\langle j_{n_x} \right\rangle$ pass through all their possible values and select those which led to the observed values of x reflection intensities. In view of the doubtful validity of the formula and the difficulties of exact measurements of intensities it is doubtful whether such a machine would justify itself at present. The method most used in practice is to reduce by means of the considerations of the last section the possible structures to a very few, and beginning with the simplest and to compare the intensities calculated from them with those observed. In this tentative method, analogies drawn from already elucidated
crystals are invaluable. In this way, proceeding step by step Brass was able to analyze rock salt, sapphire, beryl, diamond, fluorite and pyrites; all cubic crystals but of successively increasing complexity.
The Leuc method

In this method a beam of "white" i.e. heterogeneous X-rays are allowed to fall on a properly oriented crystal and the rays reflected from the various planes are registered on a photographic plate placed behind the crystal.

The reflected rays are no longer white, but consist as was shown in 9.12 of radiation made up of a series of wave lengths of different intensities.

The position of the spots on the photographic plate indicate merely the indices of the planes producing them, but as planes of the same form give spots of the same intensity, the nature of the symmetry of the crystal can be seen. The class however cannot in general be determined unambiguously, as for the reasons explained in 9.331 it is impossible to detect the absence of a centre of symmetry.

The intensities of the spots is an average effect depending partly on the spacing and densities of the planes producing them, partly on the wave length intensity distribution in the incident beam. This renders interpretation very difficult, but when the intensity distribution in the incident beam contains only one maximum, it is possible to analyse some simple crystals.

If the crystal is a lattice, the spacing for all planes is even and the reflection for that plane which best reflects wave lengths near the maximum will be the most intense. The intensities of reflections from other planes will fall off
regularly with the change of their indices. In the case of a crystal which is not a lattice however, the suppression of reflections by certain of the planes will cause the intensities to vary irregularly with their indices and these irregularities may be used to discover their structure. Even in the simplest crystals this can only be done by trying various structures with and comparing the spot intensities deduced from those observed.

In spite of these apparent limitations of the Laue method Ewald, Wycoff and others have devised improvements both on the experimental and theoretic sides by means of which remarkable results have been obtained.

A method which has occurred to me but which has not as far as I know been tried, would be to take Laue photographs with a source of homogeneous X rays of regularly varying wave length such as could be obtained for instance from a rotating rock salt crystal illuminated by white rays. If the light were strictly homogeneous, the photographs would show no spots but the central one, for all but a certain number of wave lengths for which the reflections for certain planes would flash out. If such a method were to prove practicable as full results could be obtained from it as from the Bragg method, for in this case the wave lengths being known, absolute measurements of length could be made, the absence of which is the chief drawback of the Laue method.
The Hull method

The third important method is that developed independently by Hull, and Debye and Scherrer. In this method, not one, but a large number of crystals are used in the form of a metallic aggregate or coarse powder spread on a plate or in the form of a thin cylinder. A beam of homogeneous X-rays is used. The crystals being arranged at random, there will always be a certain number that present some plane of the form \( f, g, h \) at such an angle to the incident beam that one of the orders of reflection occur and such reflected beams are detected on a photographic plate or by an ionisation chamber.

The angle that such a reflected beam makes with the incident beam is \( 2\theta \), hence if we can find \( n \), \( d_{fgh} \) is absolutely determined. It is often possible to pick out the orders of reflections due to the same plane on account of the simple relations between the sines of half the deflection angles.

The great disadvantage of the Hull method is that we have no direct means of finding \( f, g, h \). The intensities of a reflected beam is now proportional to that of the corresponding plane in the Bragg method multiplied by the number of faces in the form or in the case of a crystal not possessing a centre of symmetry, by twice this number; for it is obvious that the expectation of a suitably inclined face belonging to any form is proportional to the number of faces in the form, and that the intensity of reflection is proportional to this expectation.
It can be seen that the Hull method leads to a similar but more complicated set of equations as the Bragg method and with many more unknowns. These equations are practically insoluble, and the method accordingly used in practice is to choose the strongest of the series of reflections as corresponding to the planes with simplest indices and to check the structure thus arrived at by using it to predict the positions of the other 1 reflected beams.

The Hull method is most useful when applied to metals or to substances which can only be obtained in the so-called amorphous condition or in small and imperfect crystals.
Tabulation of results

The results obtained in an analysis of a crystal, whether or not they are complete can all be contained in the formula given in 9.34, the vectors \(a, b, c\) being also supposed known both in magnitude and in relative positions.

If however we assume the formula and also the symmetry elements of the 230 point systems as given in the tables in Ch VI all the results of an analysis may be given concisely as follows.

(i) The symbol of the skeletal lattice
(ii) the absolute dimensions both scalar and angular when necessary
(iii) the symbol of the point system. (Strictly speaking this includes the symbol of the lattice.)
(iv) the coordinates of one of each set of equivalent atoms stipulating the element.

From the set of of constants given under these heads we can calculate by the methods of Ch VII the position of every atom of each element in the crystal, and by the methods of Ch VIII the spacing and density of every set of parallel planes.

The number \(N\) of molecules of the substance per unit cell though included in the above deserves separate statement.

In the following table these structural constants are given for a number of typical crystals which have been analysed by Bragg and others.
<table>
<thead>
<tr>
<th>Name</th>
<th>Formula</th>
<th>N</th>
<th>Lt</th>
<th>Sm</th>
<th>Coordinates</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock salt</td>
<td>NaCl</td>
<td>1</td>
<td>Pr</td>
<td>O</td>
<td>Na 0 0 0</td>
<td></td>
</tr>
<tr>
<td>Zinc blend</td>
<td>ZnS</td>
<td>1</td>
<td>F</td>
<td>S</td>
<td>Zn 0 0 0</td>
<td></td>
</tr>
<tr>
<td>Diamond</td>
<td>C</td>
<td>2</td>
<td>F</td>
<td>O</td>
<td>C 0 0 0</td>
<td></td>
</tr>
<tr>
<td>Fluorite</td>
<td>CaF₂</td>
<td>1</td>
<td>F</td>
<td>O</td>
<td>Ca 0 0 0</td>
<td></td>
</tr>
<tr>
<td>Iron pyrites</td>
<td>FeS₂</td>
<td>1</td>
<td>T</td>
<td>S</td>
<td>Fe 0 0 0</td>
<td>X has been variously given as 1/5 and 2/9.</td>
</tr>
<tr>
<td>Cuprite</td>
<td>CuS</td>
<td>1</td>
<td>F</td>
<td>O</td>
<td>Cu 0 0 0</td>
<td></td>
</tr>
<tr>
<td>TIN</td>
<td>Sn</td>
<td>2</td>
<td>D</td>
<td>S</td>
<td>Sn 0 0 0</td>
<td></td>
</tr>
<tr>
<td>Rhombic alkaline sul</td>
<td>H₂SO₄</td>
<td>1</td>
<td>D</td>
<td>S</td>
<td>H₂SO₄ 0 0 0</td>
<td></td>
</tr>
<tr>
<td>sulphates</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Calcite</td>
<td>CaCO₃</td>
<td>1</td>
<td>D</td>
<td>C</td>
<td>Ca 0 0 0 0</td>
<td>Positions of atoms not given</td>
</tr>
<tr>
<td>Quartz</td>
<td>SiO₂</td>
<td>3</td>
<td>D</td>
<td>S</td>
<td>Si 0 2x/3 -x/3 -x/3</td>
<td></td>
</tr>
</tbody>
</table>

In this table no lattice constants are given.
Appendix

A Suggestion for a natural, numerical denotation of Crystal Systems, Classes, Lattices and Point Systems.

There are many systems of denotation in use. Of these the most commonly known is that of Schoenflies which is used almost exclusively for internal structure. It is essentially a literal system and in it the letters correspond to various words used to describe the systems etc. This makes it easy to learn, but it suffers from several disadvantages, each of

The basis of the system is the 32 Classes which are indicated by a capital letter with a suffix formed in general of a number and a letter. To denote 32 Classes by a triliteral symbol is unnecessary. The 14 lattices are all denoted by the letter $\Gamma$ with suffixes and dashes: here again we have a triliteral system but this one has not even the advantages of the Class notation as the manifest analogies between the different lattices are not shown by it. In the notation for point systems the method followed is to add to the Class symbol an index consisting of a serial number indicating the order of derivation of the system in the particular Class by Schoenflies' method. This order cannot claim to be a natural order, and moreover the index gives no indication either of the skeletal lattice of the system or of its affinities to similar systems.
in other classes. It is practically impossible to remember the nature of any system denoted by its Schoenflies symbol, and references to the original derivation have constantly to be made.

The system suggested is free from these disadvantages. It is entirely numerical in character, making use of one other symbol only, the decimal point, which is used to mark off the portion indicating the crystal system and class from that indicating the lattice and point system.

The symbol of a point system in its most general form consists of five numbers, two preceding the decimal point and three following. In holohedral systems, however, only one figure precedes the point and in some systems the number of figures after the decimal point often reduces to one or two. The significance of the figures is as follows:

1. The first digit from the left indicates the crystal system. The number denoting the crystal systems are as follows:

   0  Triclinic
   1  Monoclinic
   2  Orthorhombic
   3  Trigonal  (Rhombohedral)
   4  Tetragonal
   6  Hexagonal
   8  Regular  (Cubic)

It will be seen that except for the Monoclinic and Regular systems the number stands for the highest order of axes which the system possesses. We also have the convenient relation that any system is included as a special case by a system whose number is a factor of that of the first.
2. The second digit is used to distinguish the various Classes in each crystal system, so that we use a biliteral symbol for a Class.

The holohedral Classes have 0 for their second digit which can be omitted without causing confusion, thus giving the simplest symbol to the most commonly occurring Classes. The Holoaxial have 1 and so forth as shown in the table of Classes. All Classes containing only rotations of the first sort are designated by odd numbers; those with second sort rotations also by even numbers.

3. The first figure after the decimal point in conjunction with the first figure of all denotes the skeletal lattice. For normal lattices i.e. lattices with points only at the corner of rectangular parallelopipeds, this number is 0 (in the hexagonal and triclinic systems, as this is the only lattice, the figure 0 can be dropped). For centred lattices i.e. for lattices with points at the centres of rect. par. the number is 1. For face-centred lattices the number is 2. For the Orthorhombic lattice which has points at the centres of only one pair of faces of the rect. par. the number is 3. But besides these lattices, we may often consider lattices that are not distinct from those already denoted. These are \(\gamma_{0}\) for which the number is 4 and \(\gamma_{1}\) and \(\gamma_{2}\) which are denoted by 4.2 and 4.3 respectively.

In the order of lattices .0, .1, .2, it is easily remembered that each has twice the number of points per unit per rect. par. as the one before it (except in the trigonal system).

The hexagonal lattice may be denoted either as 3.0 or 6.0 according as it occurs in trigonal or hexagonal systems. The notation for lattices is given in the table.

4. The second figure after the decimal point \(\alpha\alpha\alpha\times\alpha\times\alpha\times\alpha\times\alpha\) is used to distinguish between different systems of the same Class and lattice. The number is chosen on the general idea that it is higher in systems of greater complexity. When it is 0 for instance, all the axes are rotation axes. In 4, 3, and 6, this number is the pitch of the screw axes: in 2, and 8 it represents the number of perpendicular axes. \(\alpha\times\alpha\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\times\time
As far as possible the notation of the systems is arranged so that the numbers after the decimal point are the same in analogous systems, thus indicating in a large number of cases the sub-group of any system.

The notation for the 230 point systems is given in full with the corresponding Schoenflies symbols in the table below. It can be seen that in this method of denotation each symbol contains a considerable amount of information, while at the most employing five numbers (and in this not exceeding Schoenflies) it indicates all that the Schoenflies symbols indicate and more. Moreover, it is comparatively easy to learn and with a little practice it is possible to obtain an idea of the actual configuration of all but the most complicated system by the mere inspection of its symbol.
<table>
<thead>
<tr>
<th>System</th>
<th>HOLOEHDERAL</th>
<th>Merohedral</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Holomorphlc</td>
<td>Hemimorphic</td>
</tr>
<tr>
<td>Triclinic (Oblique)</td>
<td>C, 00</td>
<td>C, 01</td>
</tr>
<tr>
<td>MONOCLINIC</td>
<td>C, 11</td>
<td>C, 12</td>
</tr>
<tr>
<td>Orthorhombic</td>
<td>Q, 21</td>
<td>C, 22</td>
</tr>
<tr>
<td>Tetragonal</td>
<td>D, 41</td>
<td>C, 42</td>
</tr>
<tr>
<td>Trigonal (Rhombohedral)</td>
<td>D, 31</td>
<td>C, 32</td>
</tr>
<tr>
<td>Hexagonal</td>
<td>D, 61</td>
<td>C, 62</td>
</tr>
<tr>
<td>Regular (Cubic)</td>
<td>0, 81</td>
<td>T, 82</td>
</tr>
</tbody>
</table>

**LATTICES**

<table>
<thead>
<tr>
<th></th>
<th>( T, 0.0 )</th>
<th>( T, 1.0 )</th>
<th>( T, 2.0 )</th>
<th>( T, 4.0 )</th>
<th>( T, 6.0 )</th>
<th>( \overline{3} ), ( \overline{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centred</td>
<td>( 3 \overline{T}', 1.1 )</td>
<td>( \overline{T}', 2.1 )</td>
<td>( \overline{T}', 4.1 )</td>
<td>( \overline{T}', 8.1 )</td>
<td>( \overline{T}', 3.1 )</td>
<td>( \overline{T}', 8.2 )</td>
</tr>
<tr>
<td>3 Face centred</td>
<td>( 3x2 )</td>
<td></td>
<td>( \overline{T}', 2.1 )</td>
<td>( \overline{T}', 4.2 )</td>
<td>( \overline{T}', 8.2 )</td>
<td>( \overline{T}', 8.2 )</td>
</tr>
<tr>
<td>1 Face centred</td>
<td>( \overline{T}', 2.3 )</td>
<td>( \overline{T}', 4.3 )</td>
<td>( \overline{T}', 2.4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In the first of each double column is given the Schonfleis serial index of the point system; in the second the decimal part of the proposed notation, the whole number part being given under the head of class.
<table>
<thead>
<tr>
<th>Class</th>
<th>1.00</th>
<th>2.01</th>
<th>3.02</th>
<th>4.03</th>
<th>5.10</th>
<th>6.11</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₄ 45</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C₄ 43</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C₄ 46</td>
<td>1.000</td>
<td>2.020</td>
<td>4.022</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C₄ 42</td>
<td>1.000</td>
<td>3.020</td>
<td>7.030</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D₄ 41</td>
<td>1.000</td>
<td>3.010</td>
<td>5.020</td>
<td>7.030</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D₄ 40</td>
<td>1.000</td>
<td>3.010</td>
<td>5.020</td>
<td>7.030</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D₄ 44</td>
<td>1.000</td>
<td>9.100</td>
<td>11.200</td>
<td>5.300</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T 85</td>
<td>1.00</td>
<td>4.03</td>
<td>5.13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tₜ 82</td>
<td>1.000</td>
<td>6.030</td>
<td>7.130</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tₜ 84</td>
<td>1.000</td>
<td>6.131</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Class</td>
<td>0</td>
<td>0.01</td>
<td>0.02</td>
<td>0.20</td>
<td>0.202</td>
<td>0.000</td>
</tr>
<tr>
<td>-------</td>
<td>----</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>35</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1.0</td>
</tr>
<tr>
<td>C</td>
<td>33</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>C</td>
<td>32</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1.0</td>
</tr>
<tr>
<td>D</td>
<td>31</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>65</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>C</td>
<td>64</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>C</td>
<td>63</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>D</td>
<td>64</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>C</td>
<td>62</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>D</td>
<td>61</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>